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Development of computer-assisted instruction units in calculus

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DEVELOPMENT OF COMPUTER-ASSISTED INSTRUCTION UNITS IN CALCULUS

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Development of computer-assisted instruction units in calculus

by

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I. INTRODUCTION

Computer-assisted instruction (CAI) is the concept of using the electronic computer to aid learning. Some equivalent widely used terms for CAI are computer aided instruction and computer based instruction. Instruction with the help of the computer is a recent educational development.

> New Trends in Mathematical Teaching (33) alludes to this, by saying:
>
> The traditional method of teaching mathematics was quite satisfactory in its day. It met the requirements and fulfilled the aims set. The period had its outstanding mathematicians and there were prominent instructors among the educators who taught this subject with excellent results. But the objectives for mass teaching then, no longer satisfy the demand now. As a result of scientific discoveries and technological development, students today face problems which only a short time ago had not even existed. These discoveries call for the increasing application of mathematics.

Hence, it may be necessary to teach mathematics in a different way to keep pace with the steadily growing demands of life and practice.

The economic and social pressures on educational institutions are forcing these institutions to improve instruction. Zinn (58) says that as the ratio of students to instructors increases, technology could be offered to assist more students without sacrificing student achievement. Public schools are being criticized for not teaching the basic skills required to pursue more advanced work. Morgan (30) notes that colleges are complaining continually that students can not read write or do mathematics. The concerns about basic skills instruction and achievement are making school systems look closely at their curriculum and instructional practices. In increasingly large numbers, educational institutions are finding that students' lack of basic skills may be remedied or eliminated by appropriate computer assistance.

The teaching of mathematics today aims at making the student capable of applying his knowledge and expanding it. In CAI, the students are encouraged to improve their capabilities. CAI allows students to make mistakes and does not subject students to continuous scolding or public censure, which might tend to decrease their self-assurance.

A. Rationale for Developing the Units

There is an increase in the use of computer technology in government, business, science and engineering. The effect of computer technology is gradually being felt in education too. In a research project, Dimas (15) states the need of developed CAI units by students in various courses.

There have been studies in the effectiveness of computer-assisted instruction in college algebra courses and in using computer-assisted instruction in overcoming attitude barriers. But there have been very few studies in developing teaching units for a calculus course in which the program is typed on a computer terminal to obtain an immediate electronic response. This study is therefore devoted to the development of computer-assisted instruction units that will facilitate the understanding of a calculus course by students.

B. Purpose of the Study

The purpose of this study is to develop a series of units with the computer being used to assist the teaching of Mathematics 121. Mathematics 121 is a calculus course taught at Iowa State University for students in engineering, science and mathematics. The CAI system chosen in this study was PLATO (Programmed Logic for Automated Teaching Operation). More information will be provided about the PLATO system in chapter four.

The general purpose of this project is to individualize instruction and create an environment in which the student's skill and self-assurance in calculus are increased. The author's experience in teaching college calculus indicates that some students have great difficulty in learning university calculus by use of the traditional lecture method. The computer might be used to enhance student comprehension of calculus. Individualized instruction means personalizing of the instructional process to conform more closely to the individual capacities of the learner. Nawaz and Tanveer (31) point out that the use of individualized instruction is a significant trend in education and represents a synthesis of philosophical and psychological thought, and is a meeting ground for instructional practice. Supported by technology and research, individualized instruction provides systematic patterns of learning for students. Some professional educators support the new idea in order to meet current demands of accountability and to improve classrom instruction and student learning.

Nawaz and Tanveer (31) state that experts view individualized instruction in a variety of ways. Some of these ways can be stated as follows:

1. Students learn through various perceptual structures. They assimilate varying amounts of content at different rates of speed and vary in their retentive abilities.
2. Learners need to develop a wide variety of learning styles for effective learning outcomes.
3. The complexity of society necessitates that students should learn on their own. The emphasis should be on the structure of knowledge and the modes of acquiring information.
4. Factors such as cost-effectiveness of education, accountability and negative effects of ability grouping provide incentives for multiple routes and alternatives characteristic of individualized instruction.
5. Learning takes place on an individual basis, and therefore educational experiences should be organized around each student. Thus the educational aspect must be flexible, adaptable and capable of meeting the demands of the individual and those of society.
6. Learning is an active not a passive process. It should involve participation in a task rather than mere absorption of information.

7. Each individual needs to develop qualities of individuality in order to cope with the complexities, and uncertainties of mass society.

The promise of individualized instruction is that of a perpetual concern for the uniqueness of each individual.

C. History of CAI

CAI was introduced in the late 1950s. Suppes and Macken (51) have divided the periods of the development of CAI into four periods namely CAI prior to 1965, CAI from 1965 to 1970, CAI from 1970 to 1975, and CAI from 1975 to the present.

1. CAI prior to 1965

Members of the computer industry were the first to start using CAI programs to train personnel in the late 1950s. At this time, electric typewriters and teletypes were linked to computers and instructional modules were presented to learners who responded with one-syllable responses. The programming languages were too complicated to be learned by the lay person, but by 1960 International Business Machines (IBM) developed the first CAI author language called Courseware 1, which educators used to program their ideas more directly.

Suppes and Macken (51) mention that in January 1963, the Institute for Mathematical Studies in the Social Sciences (IMSS) at Stanford University starteed a program of research and development in CAI which has resulted in today's widely used applications. In late 1963, IMSS demonstrated its first instructional program, which was a tutorial curriculum in elementary mathematical logic. In 1964, the preliminary version of an elementary mathematics program was tested. CAI was first used in an elementary school in 1965 when forty-one fourth grade children were given daily arithmetic drill and practice lessons in their classroom on a teletype machine that was connected to IMSS's telephone lines.

2. CAI from 1965 to 1970

During the 1965 to 1966 school year, the Stanford CAI program expanded to three schools all containing teletypes linked to the IMSS computer. At this time, about 270 elementary students and 60 high school students received drill and practice CAI lessons in mathematics while 50 elementary students continued to received the tutorial mathematical logic program at the institute.

Using an IBM 1500 system, the Brentwood Project was started. The Brentwood Project was an investigation by IMSS of the feasibility of teaching mathematics and reading as an integral part of an elementary school program by using individualized CAI over an extended period of time. Each student had two display devices, a cathode-ray tube (CRT) and a sixteen millimeter film projector of the rear projection type. Each station contained a headset for the student and a headset for the instructor, by which previously recorded oral instructions could be transmitted. The students responded by touching a light pen on one of the answer choices displayed on the CRT or by typing an answer on a keyboard.

In 1960, IMSS developed university-level computer-based programs. Stanford students in a first-year Russian program in 1967 eliminated all regular classroom work and learned at Model 35 teletypes and cyrillic keyboards and audiotapes with earphones. The statistical evaluations of this program showed positive results in terms of academic achievement and student interest level. At this time, IMSS prepared and tested an introductory college-level course in elementary mathematical theories.

At the University of Illinois another project was begun in connection with Control Data Corporation (CDC) and the National Science Foundation. This was the PLATO system which today delivers interactive material using alphanumerics, graphics, and animation. By 1971, the PLATO system had been used in curriculums concerning library use, nursing and studying CAI in role-playing games.

At the University of Texas, a CAI chemistry course was described containing fifteen modules of supplementary material for the introductory course in general chemistry while the effects of learner control in a CAI precalculus mathematics course was investigated. Also, at the University of Texas in 1970, reports were made of the results comparing computer programming with traditional instruction in the same course.

3. CAI from 1970 to 1975

Early in 1972 at Brigham Young University, there was development and field testing of the Time-shared Interactive, Computer-Controlled, Information Television (TICCIT) system of CAI. The purpose of the TICCIT project was to use minicomputer and television technology to deliver CAI lessons and mathematics to community college students. In 1974, the mathematics and English programs were being developed at Phoenix College, and a thirty-two terminal system for delivering the TICCIT program had been installed at Brigham Young University.

In 1970, the CARE project was developed. The CARE curriculum is a self-contained college level course designed to identify students with mental handicaps that are likely to adversely affect their academic progress. The method of dissemination was a mobile unit that served instructors who requested the program. Instructors in Washington, Texas, Pennsylvania and Maryland were served by this method in 1972. In 1975 the IMSS had included courses in the following languages: Old Church Slavonic, History of the Russian Literary Language, Introduction to Bulgarian, Introduction to BASIC, Introduction to LISP and many courses in music.

4. CAI from 1975 to the present

By 1978, Suppes and others at Stanford University made revisions in CAI to include elementary school curriculums and some special applications of those curriculums to hearing-impaired students. One of the

CAI systems presently available is through the Computer Curriculum Corporation (CCC). A CCC CAI consists of an instructional computer that can provide individualized lessons to as many as ninety-six CRT or teletype terminals simultaneously. The terminals are installed at an instructional site and then linked to the computer via telephone lines. CCC offers a large variety of courses for elementary and junior college students. CCC has currently several thousand terminals installed throughout the United States. Suppes and Macken (51) report that the Physics Computer Development Project at the University of California in Irvine has developed a CAI course in physics in which students control the timing of their progress and have a choice of content and method of presentation. There have been few reported developed CAI units in calculus and no complete calculus courses.

D. Outline of the Chapters

The materials presented in this study are organized into eleven chapters. The first chapter describes the need for the study, the purpose of the study and history of CAI. The second chapter includes the review of literature relating to using the computer to assist instruction. The third chapter contains a description of CAI hardware, software, and courseware and strategy for developing CAI. The fourth chapter contains description of the TICCIT and PLATO systems. The fifth chapter elaborates on the objectives of using CAI in a university level calculus course. A further discussion of the observations and discussion of the calculus units is contained in the sixth chapter. A summary of the project is presented in chapter seven. The eighth chapter contains the bibliography while the ninth chapter contains acknowledgments. Chapter ten contains the calculus units of instruction while chapter eleven contains the instructions to operate PLATO.

II. REVIEW OF LITERATURE

Mathematics 121 is a course taught at Iowa State University in which the mathematical skills of differentiation and integration are developed to a level necessary for many engineering, science and mathematics courses. The great diversity in the mathematical backgrounds and abilities of the students typically found in Mathematics 121 indicates the need for a form of instruction that can cater to the individual requirements of the students. Koistinen et al. (23) state that a great deal of evidence suggests that individualized instruction can be effective in mathematical skills.

But Stolurow (50) asserts that CAI is not the panacea for today's educational problems. There is no single solution to problems as complex as these. Computer-assisted instruction is however a substantial innovation in education. CAI has been compared to Gutenberg's invention of the printing press in terms of the potential effect it will have on education. An analysis of CAI as a concept in contrast to existing systems, suggests that it has the potential for making at least one order of magnitude of change in the educational process. The printing press made mass education possible by recording knowledge for economical dissemination. CAI makes individualization possible. The following section illustrates the capabilities and recommendations in developing CAI.

A. Attitudes towards CAI

Although many studies have compared the achievement of students using computer-assisted instruction with the achievement of students using the lecture and demonstration method, few studies have been made to assess the attitudes of students using CAI. Kockler (22) made a study of sixty-four students who enrolled in a mathematics course. The sixty-four students were randomly assigned to an experimental group or a control group. The control group received instruction by the lecture and demonstration method only while the experimental group received the same instruction usint six computer-assisted units. Both groups of students were given a pre-test and a post-test to measure their attitudes towards CAI. The conclusions of Kockler (22) were:

1. The attitude toward CAI in the experimental group improved significantly from pre-test

to post-test, but the control groups attitude toward CAI did not change significantly.

2. Attitudes toward mathematics improved in both the CAI group and the control group.
3. Computer-assisted instruction and traditional instruction both produced significant achievement gains in students, but the experimental group required less time in their instruction than the traditionally-instructed students.
4. Attitude toward CAI was unrelated to either attitude toward mathematics or achievement in mathematics.
5. Completing the short attitudinal questionnaires did not significantly affect the students' response to the major questionnaire measuring attitude toward CAI.

The growth of CAI in educational institutions depends on the attitudes of students and instructors towards CAI. Stolurow (50) says:

> Many people who first hear CAI want to see it and when they do, they go away with the feeling that it will not last. The reason is that many of the systems are not being used imaginatively. Another is that they are frequently more sophisticated internally than they are in

> terms of educational materials they display. In effect, CAI makes our meagre knowledge of teaching patiently obvious. Our ignorance cannot go unnoticed the way it does in some other form of instruction. But it would be unfair to infer that the inadequacies of the instructional program on a CAI system were the result of this type of system. This would be comparable to concluding from an observation of an ineffective teacher that all teaching should be eliminated.

B. Capabilities of CAI in Mathematics

Berkey (4) reports of an experiment at Boston University that was designed to demonstrate that the computer can be successfully integrated into the traditional calculus course. More than half of the students enrolled in a calculus course at Boston University in 1976 voluntarily participated in a programming project involving applications of the concepts to be studied during the semester. All work was done on the IBM 370/158 time sharing system whose interactive terminals were available at many locations. Students response was positive.

Many different models for the use of computers in calculus have been proposed, with none having been established as a clear preference. An approach embodied by the Center for Research in College Instruction in Science and Mathematics (CRICASAM) called for the integration of computing into a completely restructured calculus course. Another approach has been the establishment of a separate course in parallel with calculus. Yet another approach has been the using of the computer as a demonstration device or writing programs which students simply execute at time sharing terminals. The last approach has the advantage of minimizing time spent on computer work.

Berkey (4) notes that in most colleges and universities, good use can be made of the available computing facilities by integrating CAI into the traditional presentation of calculus. Students need to know what the available facilities are, how to get access to them and what the appropriate language is. Students should write their own simple programs to develop a full-fledged interest in CAI.

Using a texbook entitled "A Survey of Mathematics for College Students Using a Programming Language", Lecuyer (26) conducted an experiment to investigate the effectiveness of using the computer in teaching a mathematics course. The course was taught in two sections. A section was taught by using the above textbook and a computer while another section was taught by using the textbook alone. There was no significant difference between the performances of the two sections on the course tests, but the CAI section took less time to complete.

With these results, Lecuyer (26) concluded that the computer is rapidly becoming an extremely important and useful tool in the modern world. Lecuyer (26) also stated that it would seem that all university graduates should know something about the computer. Lecuyer (26) suggests that a computer terminal should be available to students so that they can do their homework at the terminal. The student should be presented with the complete program with an explanation of the program and an illustration of its use.

In a paper presented to the conference on computers in Undergraduate Curriculum, Day (12) talked about a course which used the computer to aid learning various mathematics concepts. In the three unit mathematics course the instructor lectured on programming concepts and served as a resource person for the final twelve weeks of the semester. The course was deemed successful since the students learned mathematical concepts well, and developed creative approaches to independent problem solving.

Kieren (21) conducted a study on the use of computers in mathematics courses, and used the computer-based drill and practice procedures in which a student interacts with a computer via one of various types of computer terminals or via a touch-telephone. Much of the work done in this field is based directly or indirectly on the work done at Stanford University. In a California study involving six grade levels in seven schools, students whose arithmetic instruction was supplemented with drill and practice, computer based instruction (CBI) programs had significantly greater post-test gains on the computation section of the Stanford Achievement Test at grades two and three, on concepts at grade three and on applications at grade six. In a similar study in Mississippi, Kieren (21) reports that there were significant differences favoring CAI. These reports indicated that apart from scholastic achievements, students did respect the computer as a teaching device, attitudes changed positively toward mathematics, but there were no reports of change in attitude toward school and no change in attendance patterns.

In 1977, a study was conducted at Copiah-Lincoln Junior College, Mississippi by Daughdrill (11) utilizing two algebra classes. One class of thirty-four students composed the experimental group, and a second class of thirty-two students composed the control group. Both the experimental and control groups were taught by the investigator. The experimental group was taught by a method designed to facilitate algorithmic thinking and problem solving by supplementing traditional classroom instruction with individual interaction with time-sharing computer terminals and with computer oriented techniques for problem solving. Flowcharts were utilized as an aid to students in analyzing and following systematic procedures in order to obtain solutions. Students were instructed to develop a flowchart which depicted a plan for solution, restate solution in the BASIC programming language and type the program on the computer terminal to obtain an immediate electronic response.

It was concluded that students who studied the BASIC programming language and solved selected algebra problems by writing and executing computer programs, performed as well on the achievement test as students who studied algebra using the traditional lecture and demonstration method. Moreover, the time spent on learning computer mathematics and using the computer as a problem solving tool, at the expense of regular classroom instruction, did not adversely affect the achievement of students in the experimental group.

C. Recommendations in Developing CAI

Sorlie and Essex (44) reported that in 1973 the school of Basic Medical Sciences at Urbana-Champaign obtained money from the federal government to develop a computer based curriculum on the PLATO IV computer system. One hundred fifty computer lessons were developed with each lesson having objectives, self-tests and multiple entry and exit points. These computer lessons were designed to be exported to other universities. From a comprehensive evaluation, it was found that students indicated preferences for those lessons which emphasized problem solving. Students liked to be able to correspond on line with the authors of the lessons.

This type of student author interaction was important since many opponents of educational computers charge that the use of a computer dehumanizes students because it removes them from exchanges with their instructors. Sorlie and Essex (44) recommend that to develop CAI, the following points should be considered:

1. A minimum of a six-month funded start-up

phase for planning and recruitment is critical to the successful development of a CAI project.

2. In order to encourage optimal computer lesson development, the project should be developed and implemented in such a way that it is an integral and meaningful part of the school's educational process.
3. Define the resources necessary to meet the objectives outlined.
4. Special attention should be given to defining the qualifications of project staff and hiring competent individuals.
5. After the staff has been recruited, project functioning should be devoted to training staff, implementing lesson development and review procedures.
6. The collection of meaningful lesson usage data requires high overhead in software and programmer resources. The collection of these data should be planned before and developed at the beginning of the project.
7. Development of a lesson usage network (multimodal-sites) seems essential, if the exportability of lessons is desired.

D. Documentation of CAI

There are many reasons why the documentation is critical in the field of CAI. Kearsley and Hunka (19) add that the development of CAI courseware is an expensive and time-consuming endeavor typically involving fifty to one hundred hours of design and programming time for each hour of instruction delivered. Clearly, this development effort can be justified if courseware can be shared by many students at different institutions. Documentation of courseware is essential to ensure that the transfer of a course from one institution to another is possible and successful. Documentation is also necessary to prevent the duplication of similar or identical courseware.

Kearsley (19) mentions two levels of documentation. The first level of documentation provides a potential user with sufficient information to determine the need for further inquiries. The first level contains the following information:

1. subject matter of the course,
2. status of the course,
3. authors of the course and their addresses,
4. availability and conditions for release of courseware,

5. characteristics of intended student group,
6. types of instructional strategies used,
7. amount of time required to complete the course,
8. support materials required,
9. the system used and the date of documentation,

The second level of documentation is designed to provide detailed information required by people who are actually working with the courseware. This level of documentation includes proctors, computer operators, instructors and students. Documentation is necessary for the ongoing continuity and stability and the instructional effectiveness of the course.

The review of literature has revealed that very few studies have been made in developing CAI units in calculus. The studies concluded that the experimental application of mathematics CAI units in elementary, secondary, and community colleges showed that students took a shorter time in completing CAI units than the lecture and demonstration method.

III. DESCRIPTION OF THE CAI SYSTEM

The traditional display of CAI is the cathode ray tube (CRT) which is similar to that used in television sets. Initial development of the PLATO project at the University of Illinois produced a plasma based display. Bork (5) says that the CRT technology appears resilient at present. Two fundamental types of information can be displayed on the screen, alphanumeric information (letters and numbers) and graphic (pictorial) information.

Bork (5) states that CAI systems can be divided into the following categories:

1. large-scale time-sharing systems in which hundreds of student stations are utilized,
2. medium-scale time-sharing systems in which fifty to one hundred terminals are used,
3. small time-sharing systems in which two to fifty student stations are utilized,
4. stand-alone systems with occasional access to a remote system,
5. pure stand-alone systems.

PLATO is an example of category one. The DEC System 10, the DEC System 20 and Sigma Series are examples of category two. The PDP 11 is an example of category four. The categories four and five are least known to students. An example of a stand-alone system is one just recently developed by Terax Corporation in Scottsdale Arizona. Another example of a stand-alone computer is the PET computer from Commodore. PET is intended for the home market and limited in its capabilities for computer-assisted learning. While hardware is important, the major issue is the production of learning materials by a wide variety of individuals.

A. The CAI Hardware, Software and Courseware

The charge often made is that CAI is dehumanizing, but Magidson (27) disagrees. The educational promise of CAI lies in its ability to individualize and personalize the instructional process and to simulate experiences not readily available. CAI lesson (courseware) can serve as text, test and tutor while compelling students to be active participants in their own learning. Students work at their own pace while the CAI lesson monitors their progress and generally prevents them from continuing to more advanced work unless mastery is demonstrated. Students have varying amounts of control over their learning in which they can review previous instruction, request special help or continue to enrichment activities. The instruction can be systematically prepared, sequenced, tested and revised.

The most basic equipment (hardware) used to deliver CAI includes a computer which stores and transmits educational material and information (courseware) by means of a specialized computer language (software). The students and instructors see the learning stations (terminals) more often than the computer. The terminal appears as a television or teleprinter which displays instruction and graphic notation and has a keyset attached to it. Students interact with the computer by means of the keyset which has the standard typewriter keys with additional special function keys. Instructors use these terminals to select curricula materials for students, to decide the sequence of these materials, to provide students with an index of lessons from which to choose, and to monitor students' progress or to prepare courseware.

CAI is usually prepared following one or a combination of the following courseware modes: drill and practice, tutorial and simulation. The drill and practice mode is the most widespread partly because it can be used to help instructors make up and check practice exercises. Typically, students are given a series of related questions to answer and are provided immediate feedback to the answers they give. As the student demonstrates mastery, more difficult questions are posed by the computer. In the tutorial mode, students are presented with instruction interpersed with appropriate questions.

Sometimes, the student is allowed to ask related questions which the computer answers. Question formats are commonly multiple-choice, matching, fill-in and short answer. Sophisticated CAI systems can catch or allow for misspellings, judge as correct a variety of possible answers including synonyms and phrases and even allow students to touch portions of the display to elicit a computer response. The simulation mode is exciting since it allows the student to discover and generate new information.

The results of a current study supported by the National Science Foundation in 1977 shows that courses in which the computer was most frequently utilized are computer science, engineering, business, mathematics, social sciences, physical sciences and education in order of the greatest usage to the least usage. In 1977, the Human Resources Research Organization (HumRRO) published an Academic Computing Directory which identified over 150 American schools, colleges and universities that have used CAI successfully. The reasons for this wide usage are:

1. evidence of student achievement,
2. evidence of increased institutional productivity,
3. a variety of applications in many subjects and courses,
4. the teaching of computer literacy,
5. an outstanding computer science or data processing program,
6. an impact on other people or institutions.

   Published studies comparing the effectiveness of CAI to traditional institution report conflicting results, but generally conclude that CAI is at least as effective and often more effective than the traditional instruction.

Since most CAI is currently being used to supplement and complement traditional instruction, not to replace it, there is great difficulty to compare CAI and traditional instruction. Magidson (27) asserts that the effectiveness of CAI is dependent upon the quality and reliability of hardware, software and courseware. An effective CAI lesson which typically takes a student an hour to complete generally takes over one hundred hours of preparation time plus student testing. The PLATO system now has over six thousand instructional hours on it. Magidson (27) also states that insofar as cost-effectiveness is concerned, the trend is toward decreasing costs for computer hardware and software despite increasing manpower costs.

A new audiovisual medium combining CAI with videodisc technology is currently being developed by the Control Data Education and WICAT Inc. The videodisc combined with a microprocessor, will permit motion pictures to be used in an interactive mode. The motion picture can be viewed in single frames, in slow motion or reverse order sequence without sound. Graphic and textslides can be interspersed with motion pictures in any course mode using sound, words, graphics and animation. This new audiovisual medium will enhance the systems approach to instructional design and development.

B. Basic Skills Instruction Using CAI

Computer support is essential for keeping detailed records on student achievement and for determining both individual and group progress through the curriculum. Morgan (30) proposes that if an institution of learning decides to purchase, lease or develop its own CAI, the following aspects should be considered.

1. Entry level

To determine the suitability of a program for an individual student, the CAI module might provide an on-line entry level skills test. The instructor should have a list of objectives on which a student must show mastery before beginning the new program.

2. Objectives

Instructors should be provided with the objectives on each CAI program. The objectives should be organized for instruction and each category sequenced by difficulty level.

3. Strategy

The development of a CAI program should show how the instruction is presented. It should be known whether the method of presentation is drill and practice, tutorial, simulation or game.

4. Criteria

Information about the successful completion of the total program must be documented.

5. Individual needs

The CAI strategies should be analyzed to determine their adaptability to individual needs. When additional instruction is necessary, the techniques of the second presentation should be different in clearly identifiable ways from the original instruction. Branching should occur automatically whether the student needs enrichment, acceleration or remediation and students should be able to interact with CAI independently.

6. Reinforcement

Students sometimes work on paper and pencil tasks with no knowledge of how well or poorly they are doing, but CAI allows each exercise to be graded as it is completed. The reinforcement of knowing how well you are doing immediately upon completion of the task, is one reason students like CAI.

7. Feedback

Not only is each problem or exercise graded or recorded, but aggregated information is available on an objective or a set of objectives. CAI can operate in some ways as a private tutor.

8. Diagnostics

CAI makes individual records available to the instructor. Upon the student's completion of an interactive CAI session, the instructor can find out what the student has worked on and what should be done next.

9. Student involvement

CAI demands active involvement from the student. Creative programs motivate students. Although a student may enter an answer after a little or no thought, a response of "wrong" from the computer is unwelcome. After the first few impulsive answers, most students reflect before responding. The result of continuing to input thoughtfully will give some encouragement to do better.

10. Validity and acceptability

CAI programs must teach what they purport to teach. Students and faculty should react favorably toward the use of CAI, and educators should perceive its instructional value.

11. Efficiency and effectiveness

Studies report that the mean time for course completion using CAI is about one-half to two-thirds of the standard time alloted to the course. Morgan (30) adds that CAI drill and practice mode is consistently effective and that CAI is effective as a supplement to instruction rather than a substitute for instruction. The need of instructors and specialists for achievement data on specific objectives and for direct assistance with instruction makes the computer a natural ally for instruction.

C. Relationship between CAI and CMI

Instructional utilization of computers is usually subdivided into two categories, namely, computer-assisted instruction (CAI) and computer-managed instruction (CMI). Splittgerber (48) defines CAI as a teaching process directly involving the computer in the presentation of instructional materials in an interactive mode to provide and control the learning environment for each individual student.

CMI is the instructional management system utilizing the computer to direct the entire instruction. The distinct difference between CAI and CMI is that in the CAI mode, the computer functions as a teacher while in the CMI mode the computer functions as a manager.

In practice, computer instruction does not dictate the type of instruction found in universities. In figure 1, CAI is illustrated as focusing on the direct teaching of concepts and skills while CMI is illustrated as being a broad concept encompassing the typical modes included in CAI and other forms of instruction which do not directly require the use of the computer. Since CAI does not have to be employed in order to have CMI implemented but CMI is usually required to manage CAI data generated by students, the CAI rectangle is separated by an arrow.

TRADITIONAL NON-COMPUTER-ASSISTED INSTRUCTION

Any traditional non-computer teaching and/or learning strategy including:

- Lecture
- Group activities
- Question/answer
- Learning centers
- Laboratory instruction
- Experiential community based education

COMPUTER-MANAGED INSTRUCTION

An instructional management system involving

- Organizing Curricula and student data
- Monitoring student progress
- Diagnosing and prescribing
- Evaluating learning outcomes
- Providing planning information for teachers

↕

COMPUTER-ASSISTED INSTRUCTION

A teaching process including any one or more of the following:

- Drill and practice
- Tutorial
- Simulation and gaming
- Problem-solving

Figure 1. Relationship between CAI, CMI, and traditional instruction.

Splittgerber (48) reports of the following conclusions about CAI:

1. Generally, CAI has the potential to be an effective instructional aid when measured through the results of student achievement. It appears to be more effective in tutorial and drill modes, and also more effective for low-ability students than for middle or high-ability students.
2. When students are permitted to proceed at their own rate, they will generally learn more rapidly through CAI than through traditional instructional methods.
3. The retention rate of material learned under CAI appears lower than for traditional instructional approaches.
4. CAI is as effective as other means of individualized supplemental instruction if it is utilized as a supplement to regular classroom instruction.
5. Despite equipment malfunctions, students are highly enthusiastic about CAI as an instructional mode.

In table 1, are data on presently used major CAI and CMI systems.

Table 1. Data on the major CAI and CMI systems.

PROJECT	DEVELOPER	SERVICES
Automated instructional Management System (AIMS) TYPE - CMI	New York institute of Technology	Evaluation of student progress, prescriptions, and empirical validation and optimization of instruction
Wisconsin System of Instructional Management (WIS-SIM) designed for Individually Guided Instruction (IGE) TYPE - CMI	Wisconsin Research and Development Center for Cognitive Learning	Criterion-referenced tests, achievement profiling, diagnosis, prescription, and instruction
Individually Prescribed Instruction/ Management Information System (IPI/MIS) TYPE - CMI	Learning Research and Development Center at University of Pittsburgh assisted by Research for Better Schools	Diagnoses and prescribes. Collects and processes information, competence, performance, and progress of each student
Instructional Management System (IMS) TYPE - CMI	System Development Corporation Southwest Regional Lab	Assists with pacing, grouping sequencing, and individualization
Program for Learning in Accordance with Needs (PLAN)	American Institutes for Research and now managed by	Monitoring and supervising, test scoring, diagnosis, prescription,

Table 1 continued

PROJECT	DEVELOPER	SERVICES
TYPE - CMI	Westinghouse Learning Corporation	individualization, inservice for teachers
Interactive Training System (ITS) TYPE - CAI	International Business Machines Corporation	Develops courses, teachers write courses, any mode of presentations is permitted
Programmed Logic for Automatic Teaching Operation (PLATO) TYPE - CAI	University of Illinois, now managed by Control Data Corporation	Any CAI mode can be employed, develops courses and units, especially helpful with simulations and game playing, revision and editing at any time
Stanford Project TYPE - CAI	Stanford University	Revisions and editing at any time, but for problem-solving and drill-practice, schools usually contract for services
Time-Shared, Inter-active, Computer-Con-trolled, Information Television (TICCIT) TYPE - CAI	MITRE Corporation	Revision and editing of program, monitoring, and evaluating student progress utilizing all four modes of CAI

D. Strategy for Developing CAI

A team approach is desired to develop courseware involving content specialists, instructional designers, programmers and evaluators. Dimas (15) suggests the following process of developing courseware:

1. The team members of a given discipline meet for the purpose of determining curricula and lesson priorities. During this first step, the areas within the course are analyzed in order to determine where students are experiencing the greatest difficulty. These areas are assigned a high priority for lesson development, and a decision is made as to whether CAI can alleviate the learning problem or not.
2. Assignments are undertaken by the faculty for the development of a one or two page scenario. A scenario is an overview of a lesson and should include student objectives, a description of the content, and a pedagogical approach.
3. Scenarios are submitted to team members and criticized in a group setting. Revisions, which are agreed upon by the team, are incorporated

in the scenario. The use of discussion does much to eliminate false starts on a lesson script, which is a frame by frame hard copy view of the lesson. The faculty member uses a full sheet of paper to simulate a frame of CAI. Answer judging and other information appear at the bottom of the page.

4. A lesson script is developed from the scenario and submitted to the team where it is critiqued in a group setting. Revisions are made and the script is again discussed by the group. The process is repeated until the script is approved by the team members. It should be mentioned that the discussion provides many worthwhile suggestions and a feeling of trust both of which usually contribute greatly to the quality of courseware.
5. Upon approval of the lesson script, a computer file space is assigned and a programmer is provided.
6. The precise instructions needed to present the lesson script are entered into the computer by the programmer.
7. The lesson is criticized by the team as portions of the script are programmed. Once

complete, a written critique of the lesson is submitted by each member and discussed in a group setting.

8. Revisions which have been agreed upon by the group are incorporated into the lesson by the author and programmer.
9. The lesson is tested with a few capable students, data are recorded, and revisions are made.
10. The lesson is student tested and objective data are obtained.
11. Using the objective data gathered from the previous step, the lesson is again revised as needed. This step is repeated as many times as necessary until the lesson is complete and published.

Even though the team approach to courseware development appears to have the greatest potential, Dimas (15) proposes that individual effort in the authoring or programming process should be nurtured wherever possible because:

1. The programming abilities of faculty and staff will be extended.
2. An atmosphere of creativity will be fostered

which will bring about more effective forms of CAI.

3. Faculty who have commitment to instructional improvement and who may become advocates of CAI can be identified.

4. Content areas where potential users exist can be identified, provided courseware is made available.

5. A mechanism for examining new and unorthodox means of instruction which may have a high payoff in terms of student achievement is provided.

6. A means of fulfillment for those individuals who would like the challenge of starting and completing the entire process is provided.

The system of courseware development in any institution should allow faculty and staff to become a part of a creative and stimulating process.

IV. THE TICCIT AND PLATO SYSTEM

Rhw PLATO IV system is probably the largest, most heavily funded CAI experiment in the world. Denenberg (13) reports that the original National Science Foundation (NSF) grant of five million dollars and the matching five million dollars from the State of Illinois have, since 1959, resulted in a veritable river of software and about one-half dozen hardware sites capable of supporting that software. Two best known approaches will be discussed in this chapter. They are the Time-shared, Interactive, Computer-Controlled, Information Television (TICCIT) system and the Programmed Logic for Automatic Teaching Operations (PLATO).

A. The TICCIT System

The TICCIT system was a small CAI facility which combined minicomputers and television receivers in an instructional system with the display capabilities of color television. Alderman et al. (1) report that teams of specialists were assembled to produce courseware which aimed at providing a complete and independent alternative to entire colege courses in selected subjects.

1. Systems description of TICCIT

The TICCIT system uses the capabilities of mini-computers with the strengths of television receivers. By relying on mini-computers and other equipment already available for purchase commercially, the developers of TICCIT have taken advantage of proven technology and kept the costs low. Two Nova 800 discs from Data General along with a bank of discs for memory and data storage, enable each TICCIT system to serve one hundred twenty-eight active terminals. The terminal is a color television set modified to accept digital computer signals and translate them into display frames. Students use the electronic keyboards that accompany the television receivers to interact with the computer system. Further, students direct their own instruction through TICCIT's unique approach to instructional design. Options built into the TICCIT system permit students to exercise control over their own instruction.

Students choose a unit, a difficulty level appropriate to their own performance, and an instructional sequence for learning.

2. Context of the demonstrations

The TICCIT program became an integral part of the curriculum at some universities. Denenberg (13) reports that in these institutions, students could register for courses and even earn college credits in classes taught primarily by the computer. Instructor involvement varied from direct supervision of all student work to supplementary assistance provided upon student request. Instructors in mathematics courses where the department policy set the TICCIT coverage according to curriculum requirements, had responsibility for managing classes sometimes three times the size of usual lecture sections and for advising students on their course progress. All courseware followed the same instructional design, essentially a form of learner control built around hierarchical content structure.

Participants in this study also developed a comprehensive plan for implementing the TICCIT program and a manual for introducing faculty to the system. The TICCIT program depended on college faculty to determine its content structure and to revise the initial versions of courseware.

3. Effects of the TICCIT Program

In the 1975-76 academic year, Alderman et al. (1) stated that in an evaluation of five thousand students in nearly two hundred sections of target courses, the TICCIT program offered a great opportunity to contrast the results of courses taught by computer with the results of conventional practices. The evaluation concentrated on four aspects of performance namely, course completion rates, student achievement, student attitudes and student activities.

The impact of the TICCIT program on course completion rates, defined as the proportion of students enrolled in a course who later fulfill the course's requirements and receive grades with credit was negative in every case except one. For instance, the average completion rate for mathematics courses was 16 percent for TICCIT classes and 50 percent for lecture sections. It appears that the TICCIT program had detrimental effect on the likelihood that a student would complete the college requirements for course credit. Students stayed with the program, but failed to complete all the lessons required in order to earn college credits.

Students completing a mathematics course on the TICCIT system had higher post-test scores than comparable students in lecture sections. Estimates of the size of significant TICCIT effects indicated an increase over the achievement outcomes of conventional mathematics instruction. Student reactions to the TICCIT program were generally favorable. However, viewed in comparison with student attitude toward classroom teaching methods, the affective outcomes of the TICCIT program were often less positive. In mathematics courses, students rated special features of the TICCIT program lower than the classroom counterparts. Further, students in lecture sections reported greater satisfaction with the amount of individual attention given them, than did students in TICCIT classes.

To a great extent, this may have resulted from the high student-instructor ratio in some TICCIT classes. With a class of comparable size consisted with traditional practices, the attitudes of students toward learning on the TICCIT system were about the same with those in lecture sections. Indeed, English classes on the TICCIT system supplemented by small discussion groups with an instructor led to more positive student attitudes than did lecture-discussion classes. Perhaps the original specifications for the TICCIT program had underestimated the importance of the instructor to students.

In considering the impact of the TICCIT program on course completion rates, it was found that any form of self-pace instruction is likely to exert a negative effect on the pace of student learning. The completion rates in TICCIT mathematics classes were comparable to those found in other sections taught by programmed instruction. Analyses for achievement results and course completion rates suggested that those students who benefited most from learning on the TICCIT system, were students who were stronger in their subject matter preparation with students of a similar background in the subject matter, Alderman et al. (1) suggest that there might be consistent positive results on all aspects of student performance.

B. The PLATO System

The PLATO (Programmed Logic for Automatic Teaching Operations) system is the CAI system the author has chosen to develop university level calculus units. The PLATO System is based at the University of Illinois and supports nearly one thousand terminals at dispersed locations and provides each site with a central library of lessons. The use of lessons in the PLATO system depends on its attractiveness to teachers and students, and appropriateness to specific courses.

The PLATO computer-based education is the largest and perhaps most sophisticated computer system designed for education. It has one thousand terminals connected to a Control Data Corporation Cyber 73-24 computer based in Urbana, Illinois. The heaviest concentration of terminals is in Illinois, but there are many terminals throughout the United States and some in other countries.

Communication between the central computer and dispersed terminals occurs over telephone lines for distant sites, and microwaves for nearby locations. The display screen for a PLATO terminal is a plasma panel. Backed by the computing power of the central computer, estimated at four million instructions per second, the panel can relay dynamic graphics and thereby perform such tasks as illustrating principles in the physical sciences or simulating laboratory experiments. Lessons also include repetitive drills giving students practice in basic concepts. Students indicate input messages through keysets similar to those on electric typewriters. Instructors also use PLATO terminals as they develop lessons written in a special author language called TUTOR.

Each site for a PLATO terminal can give access to any PLATO lesson stored in the central library. In addition, any user of the PLATO system can communicate directly with any other user on the system.

1. Context of the demonstrations of PLATO

Alderman et al. (1) reported a study which was conducted in five community colleges in Illinois with a total of 116 terminals. Although PLATO lessons were available in many different sunject areas, the primary thrust of the study was concentrated in accounting, biology, chemistry, English and mathematics. Instructors in the colleges determined how much the system would cost and what lessons would be available to them. The autonomy permitted instructors, was consistent with the developers' goal of making a powerful resource available for education. Generally, instructors integrated PLATO lessons into their class curriculum and replaced portions of their own classroom coverage. Even though the PLATO system had anticipated that instructors would develop PLATO lessons by themselves, it soon became necessary for a central staff to coordinate teacher efforts and so avoid redundancy and facilitate dissemination among the colleges. Alderman et al. (1) state that PLATO developers had overestimated the proportion of a course that would be taught on the system, for instance, less than one-third of student instruction was given in the CAI mode.

2. Effects of the PLATO System

Alderman et al. (1) also mentioned that the PLATO system provided instruction to approximately four thousand students in each semester of the study. Because many instructors agreed to teach a section of a course with PLATO lessons as well as another section of the same course without student exposure to the PLATO system, it was possible to implement a design which was partially balanced for possible instructor effects and college effects.

The basic finding of the evaluation in the areas of student attrition and achievement was neutral. Student exposure to the PLATO system had no consistent impact on attrition. The average completion rate for PLATO answers was 58 percent in contrast to an average of 59 percent for non-PLATO courses. Among twenty-three populations examined for achievement, there were eleven positive PLATO effects and twelve negative PLATO effects. The few significant effects for either outcome could be plausibly explained by instructor differences.

The impact of the PLATO system on student attitudes was generally favorable. PLATO students showed significantly more favorable attitudes toward computer-assisted instruction than non-PLATO students. Seventy to ninety percent of the students liked the fact that they could make mistakes without embarrassment, that PLATO made helpful comments on their work, that PLATO make good use of examples and illustrations, that they could take part in their instruction at each step in the lesson, and expressed the desire to take another course in the PLATO system.

Eighty-eight percent of the students disagreed that using PLATO was dehumanizing or boring. In the comparisons of PLATO and non-PLATO students, equally large percentages of both groups of students felt challenged to do their work, thought that they received individual attention, felt free to ask questions and express opinions, often discussed their course material with other students, did not find it difficult to get help when they did not understand the material in their courses, and would recommend their respective courses to their friends. The results tend to refute some common stereotypes that computer-assisted instruction may have an isolating effect on students. Alderman et al. (1) reported that in 177 observations of PLATO sessions, trained Educational Testing Service (ETS) observers noted an increase in contact between students and instructors and between students themselves. The observers rated the students as generally very attentive to their work, relaxed, enthusiastic, and active with the PLATO system.

The impact of the PLATO system on instructors was a favorable one. For instance, 72 to 86 percent of the instructors judged the number and content of PLATO lessons, the clarity of the material presented and the use of examples and illustrations to be adequate for their students. Instructors did not perceive the PLATO system as isolating the students from them. In fact, 39 percent of the instructors thought that they had more contact with the students because of PLATO, while only 15 percent thought that their contact with students was decreased because of PLATO. About 78 percent of the instructors did not perceive the use of PLATO as decreasing their workload and 39 percent thought that their workload was increased because of PLATO. Eighty-eight percent of the students intended to continue using PLATO in their courses. About 80 to 83 percent of the instructors judged PLATO to have a positive impact on student achievement.

The wide acceptance of the PLATO system without appreciable negative impact on student performance seems consistent with the conditions under which the study took place. There may have been too little time spent on the PLATO system to affect student achievement in an entire course. In most cases, students spent less than eight hours on the the PLATO system for a course. A number of alternative explanations might account for student's attraction to the system such as the novelty of CAI or the sophistication of the PLATO system.

C. Evaluation of the PLATO System

The following are negative aspects of PLATO:

1. PLATO has some hardware and software problems that decrease its effectiveness. The usage of the many terminals at once can cause the system to stop. The Control Data Corporation (CDC) is remedying the situation.
2. A central pedagogical philosophy within the PLATO system is that anyone can teach himself TUTOR and be busily writing lessons for students within a short time. This philosophy

may hold for a very simple CAI program, but the more useful lessons require a degree of expertise not commonly found in many professional programmers.

The positive aspects for which PLATO was chosen as a CAI system to assist in developing the twenty CAI units are:

1. The PLATO system allows individualized instruction. On the PLATO system, each student can proceed at his own speed through the sets of units that comprise parts of the course.
2. PLATO CAI is especially fruitful in the area of drill and practice. Mathematics and many areas in the physical sciences seem to fit the PLATO format.
3. The library of existing courseware (programs) is impressive even though there are few CAI units in mathematics at the university level. The special function keys allow rapid, creative and enjoyable composition of lessons by authors.
4. The interest in the system does not wear off after a long period of time.

5. Income from royalties and licensing fees being paid by commercial users of inventions developed for the PLATO system are now providing a substantial return on the investment in PLATO.

Like any other man-made tool, PLATO technology can be used or misused with equal ease. PLATO's potential is deep and broad and if intelligently and humanely used can help people educate themselves.

V. DESCRIPTION OF THE CALCULUS UNITS

The first step in the development of the calculus units as illustrated in chapter three, section D, is a preparatory stage which is divided into the following categories:

1. choosing a topic,
2. stating the course objectives,
3. writing the content outline,
4. stating the instructional objectives,
5. obtaining pre-test scores.

Since a discussion of categories 1 and 5 is contained in the third chapter, this chapter will be concerned with categories 2, 3, and 4. In section A, categories 2 and 3 will be discussed while in section B, category 4 will be discussed.

A. General Objectives

Since Mathematics 121 is a course taught in Iowa State University, the selection of the subject matter in the course has already been made by the mathematics department. There are two instructional objectives for this course. The first instructional objective is concerned with:

1. creating new and concrete situations that enhance the learning of the material,
2. introducing the students to a more advanced course Mathematics 122 and assisting the students in their applications to their disciplines.

The second instructional objective is that of providing the student with an understanding of calculus frequently employed in mathematics, engineering and the physical sciences such as

1. applying the first and second fundamental theorems of calculus,
2. computing integrals by using formulas,
3. integrating by trigonometric substitutions, by parts, completing the square, by partial fractions, and by powers of trigonometric functions,
4. computing cross-sectional length, cross-sectional area, improper integrals, polar coordinates,
5. writing parametric equations,
6. calculating area in polar coordinates, surface area, and volume of revolution.

These two instructional objectives not only describe, but also assist in analyzing the tasks that the student is to perform. A task analysis of the goals is conducted by defining the prerequisite behaviors necessary for the student to attain. The identification of each unit of instruction completed the course outline, and helped in defining the objectives of the course.

B. Objectives of Each Unit of Instruction

A task analysis of the twenty instructional units showed more sub-tasks. The following information contains descriptions and instructional objectives of the twenty instructional units.

In Unit 1, the first fundamental theorem of calculus pertains to the integration of continuous functions in closed intervals. The function could be expressible in terms of polynomials, logarithms, exponentials, trigonometric functions, or any composition of these functions. There is one problem already solved for the student and five problems in which the student is required to fill the blanks to be able to solve the problems.

Upon completion of this unit of instruction, the student should be able to:

1. compute integrals of continuous function in a closed interval,
2. differentiate polynomials, logarithmic functions, exponential functions, trigonometric functions and any composition of these functions,
3. solve all the twenty-five problems at the end of this unit of instruction.

The second unit continues with the second fundamental theorem of calculus by differentiating an integral. A continuous function $f(x)$ is defined on a closed interval $[a,b]$. Then a student is given a new function $y(x)$ and defined as equal to $\int_a^x f(t)\,dt$ for $a \leq x \leq b$. The function $y(x)$ is said to be differentiable and its derivative is $f'(x)$. There are two problems already solved for the student and five problems in which the student is required to fill the blanks to be able to solve the problems.

Upon completion of this unit of instruction, the student should be able to:

1. differentiate integrals,
2. set up the function to be differentiated,
3. solve all the twenty-five problems at the end of this unit of instruction.

Up to the third unit, the method of evaluating has been by the fundamental theorems of calculus. In this unit, eleven formulas will be used to evaluate integrals of common functions. There are two problems already solved for the student and five problems in which the student is required to fill the blanks to be able to solve the problems.

Upon completion of this unit of instruction, the student should be able to:

1. select the right formula to use for each specific problem,
2. recall all eleven formulas,
3. solve all the twenty-five problems at the end of this unit of instruction.

The fourth instructional unit deals with computing integrals of exponential and rational functions. In this unit, attention is given to the integration of three types of functions. There is one problem already solved for the student and five problems in which the student is required to fill the blanks to be able to solve the problems.

Upon completion of this unit of instruction, the student should be able to:

1. select the right formula to use for each

specific problem,

2. solve all the twenty-five problems at the end of this unit of instruction.

The fifth unit is concerned with integration by trigonometric substitutions. Some ten formulas are provided and three illustrations of how three dominant integrands are used in computing integrals. There are three problems in which the student is required to fill the blanks to be able to solve the problems.

Upon completion of this unit of instruction, the student should be able to:

1. select the right formula to use for each specific problem,
2. draw appropriate diagrams to facilitate problem solving,
3. solve all the twenty-five problems at the end of this unit of instruction.

The sixth unit considers the method of integrating by parts. The derivation of the method of integration by parts is shown. There is one problem already solved for the student and five problems in which the student is required to fill the blanks to be able to solve the problems.

Upon completion of this unit of instruction, the student should be able to:

1. separate the integral into two appropriate parts,
2. choose one part to be easily differentiable and the other part to be easily integrable,
3. solve all the twenty-five problems at the end of this unit of instruction.

In the seventh unit, completing the square and integration by completing the square is emphasized. The general quadratic function is rewritten to show the method of completing the square. There is one problem already solved for the student and three problems in which the student is required to fill the blanks to be able to solve the problems.

Upon completion of this unit of instruction, the student should be able to:

1. complete the square,
2. use an appropriate formula in a previous unit of instruction,
3. solve all the twenty-five problems at the end of this unit of instruction.

In the eighth unit of instruction, the method of integration by partial fractions is provided. The definition of polynomial and rational function is mentioned. There is one problem already solved for the student and four problems in which the student is required to fill the blanks to be able to solve the problems.

Upon completion of this unit of instruction, the student should be able to:

1. select the appropriate case for the problem,
2. complete the square,
3. solve all the twenty-five problems at the end of this unit of instruction.

The ninth unit of instruction is concerned with integration of powers of trigonometric functions. The unit lists identities useful in integrating powers of trigonometric functions. There is one problem already solved for the student and four problems in which the student is required to fill the blanks to be able to solve the problems.

Upon completion of this unit of instruction, the student should be able to:

1. select the right identity or combination of identities to use,
2. convert from one identity to another,
3. solve all the twenty-five problems at the end of this unit of instruction.

The tenth unit of instruction pertains to integration by miscellaneous substitutions. In this unit, four types of substitutions have been provided. There is one problem already solved for the student and three problems in which the student is required to fill the blanks to be able to solve problems.

Upon completion of this unit of instruction, the student should be able to:

1. choose the correct substitution,
2. change integrand to the appropriate form,
3. solve all the twenty-five problems at the end of this unit of instruction.

The eleventh unit of instruction deals with computation of cross-sectional area. The unit develops the arc length formula in three forms. There is one problem already solved for the student and three problems in which the student is required to fill the blanks to be able to solve the problems.

Upon completion of this unit of instruction, the student should be able to:

1. select the correct formula to use for each specific problem,
2. differentiate appropriately,
3. substitute in the formula,

4. solve all the twenty-five problems at the end of this unit of instruction.

In the twelfth unit, computation of cross-sectional area is emphasized. The unit develops a method of calculating the cross-sectional area by the use of an elementary strip. There is one problem already solved for the student and three problems in which the student is required to fill the blanks to be able to solve problems.

Upon completion of this unit of instruction, the student should be able to:

1. make a sketch of the function,
2. draw an elementary strip,
3. write the area required,
4. solve all the twenty-five problems at the end of this unit of instruction.

The thirteenth unit of instruction is an enrichment pertaining to the average of a function over an interval. This unit presents the notion that the area of the rectangle is equal to the area under the graph. There is one problem already solved for the student and three problems in which the student is required to fill the blanks to be able to solve problems.

The student, after completing this unit of instruction, should be able to:

1. write the difference between the intervals,
2. compute the required integral,
3. solve all the twenty-five problems at the end of this unit of instruction.

In the fourteenth unit, improper integrals are emphasized. This unit of instruction deals with two cases of functions that are continuous at a point. There is one problem already solved for the student and three problems in which the student is required to fill the blanks to be able to solve the problems.

After completing this unit of instruction, the student should be able to:

1. select the case to use,
2. write the limits of integration,
3. solve all the twenty-five problems at the end of this unit of instruction.

In the fifteenth unit, polar coordinates are introduced. This unit gives a method of changing from one coordinate system to another. One problem is solved for the student and three problems in which the student is required to fill the blanks to be able to solve the problems.

Upon completion of this unit of instruction, the student should be able to:

1. convert from rectangular coordinates to polar coordinates and vice-versa,
2. change polar equations to rectangular equations and vice-versa,
3. solve all the twenty-five problems at the end of this unit of instruction.

The sixteenth unit of instruction is concerned with infinite sequences and series. This unit includes definitions on sequences and sums, and a proof of the convergence of geometric series. There is one problem already solved for the student and three problems in which the student is required to fill the blanks to be able to solve the problems.

Upon completion of this unit of instruction, the student should be able to:

1. compute sum of a convergent series,
2. find out if a series is convergent or divergent,
3. solve all the twenty-five problems at the end of this unit of instruction.

Unit seventeen continues with the area of polar coordinates. This unit of instruction describes a method of using a sketch to evaluate the area in polar coordinates. There is one problem already solved for the student and three problems in which the student is required to fill the blanks to be able to solve the problems.

After completing this unit of instruction, the student should be able to:

1. sketch the appropriate diagram,
2. write the area of a sector of a circle,
3. solve all the twenty-five problems at the end of this unit of instruction.

In the eighteenth unit, the area of a surface of revolution is examined. This unit deals with four different formulas for calculating the area of a surface of revolution. There is one problem already solved for the student and three problems in which the student is required to fill the blanks to be able to solve the problems.

After completing this unit of instruction, the student should be able to:

1. select the appropriate formula for the problem,
2. draw a sketch of the area required, if necessary,
3. solve all the twenty-five problems at the end

of this unit of instruction.

Unit nineteen pertains to the volume of a solid of revolution. This unit deals specifically with the disc and shell methods of finding the volume of a solid of revolution. There is one problem already solved for the student, and three problems in which the student is required to fill the blanks to be able to solve the problems.

Upon completion of this unit of instruction, the student should be able to:

1. sketch an appropriate volume,
2. write the integral of the volume,
3. solve all the twenty-five problems at the end of this unit of instruction.

The twentieth unit of instruction is the final examination which consists of all the previous nineteen units. At this final stage, the student should be able to solve all the twenty-five problems provided at the end of this unit of instruction.

VI. DISCUSSION

The CAI units of instruction in calculus described in the preceding chapter can be used in a calculus class in addition to the lecture and demonstration. The twenty units of instruction should be learned in one quarter. However, the option is provided for self-pacing, which allows for the completion of the instructional units according to a student's schedule or motivation. The student can ask questions from the instructor or refer to the class text or any additional supplemetary material.

Many of these ideas, together with observations in this study are discussed in the following section of this chapter. Recommendations for future development of these CAI units on calculus are reflected in section B of this chapter.

A. Observations

The present CAI systems such as TICCIT and PLATO will be superseded by current revolutions in large integration and videodisc which will be improved through new generations of technology for education. Dwyer (16) states that the complexity inherent in human nature drives the relationship between technology and education. That deep technology is of no importance without a deep view of education.

From the review of literature, it was found that computer systems have been designed to generate equivalent test forms. Therefore, the question is not whether the computer can be used in this way, but, rather, how computer systems can be designed to meet the particular needs of an individualized mathematics program and to enhance the effectiveness of this instruction.

The CAI units in this study consist of the main topics of the course. The first CAI unit was programmed in a PLATO terminal at Iowa State University. About six hundred hours were required to complete programming this CAI unit, while only five hours were required in the preparation of this unit by the method of lecture and demonstration.

Nearly all the students who were introduced to this CAI unit of instruction appeared highly motivated. Some of the students were so interested in this unit that they completed a unit within three days. The more capable students appeared to enjoy CAI because it allowed self-pacing which is lacking in many lecture and demonstration methods. Another positive feature that was mentioned by the students was the feedback they received during their interaction with the comuter. Student interest was shown when there was a breakdown of the computer. They continually inquired when the system was to become operational.

Suggestions from students about portions of the program that could be misunderstood were valuable. Through the analysis of student responses, clarifying statements were included. The amount of time required to complete a unit of instruction ranged from 50 to 120 minutes, while 45 to 100 percent of the scores were correct. A major limitation encountered in this study was the considerable time required to program the relatively small amount of student contact time at the terminal.

The CAI system used in this study, PLATO, was developed when the resources of educational, commercial, and government organizations were combined. This development had an outcome rare in the history of government support of education. Income from royalties and licensing fees being paid by commercial users of inventions developed for the PLATO system are now providing taxpayers with a substantial return on their investment in addition to long-term educational benefits. Stolurow (50) reports that PLATO lessons now show steady improvement in instructional effectiveness.

B. Recommendations

The future of CAI is questionable. On one hand, there is restriction of funds. For example, there are no new projects whose funding is comparable to the funding by the National Science Foundation of the TICCIT and PLATO projects. On the other hand, the commercial sector is developing which means more restricted research. The technological trend toward miniaturization will tend to reduce unit cost due to "chip" technology. This relatively low cost will enable students in many academic disciplines to take advantage of the new powerful tools. If the semiconductor industry keeps up the trend of providing denser memories at a lower cost per bit, then more attractive mini CAI systems will be created so that reliable, secure, and transferable software and courseware are produced.

The first recommendation is that, before the use of PLATO or any CAI system, the student should be well-informed of the capabilities of the CAI system being utilized. The student should be familiar with sign-on procedure, the terminal's keyboard, and the computation mode.

The second recommendation is that these CAI units in calculus will be tested in many calculus classes. Since an effective CAI program requires periodic content updating and expansion of individual features, the curriculum material should be continually changed. These further tests can assist the CAI program.

The third recommendation is concerned with the operational management of the PLATO system. There is a definite power hierarchy from student, author, course director to the PLATO project programmer. To be successful, all members must cooperate so that there will be coordination at each step of the process. Although this hierarchical power structure seems to be necessary to ensure some level of security, it should not promote elitism.

Finally, CAI has good capabilities in individualizing instruction, doing research on various teaching modes, and developing ways of assisting instructors and authors in the development of instructional materials. If CAI is intelligently and humanely used, it can really help people educate themselves.

VII. SUMMARY

The primary purpose of this study was to develop computer-assisted instruction units in calculus for students at the university level. These CAI units were designed to provide instruction and related practice problems to mathematics, engineering and science students who are enrolled in Mathematics 121 at Iowa State University. The emphasis of this study was on the development of operational CAI units in calculus.

A description of the three main components of a CAI system namely the hardware component, the software component, and the courseware component was given. The facilities of the PLATO terminals at Iowa State University were used for this study. The PLATO system contains a programming mode which an author can use in programming lessons or in conversing with an author.

Since the computer-assisted units in the study are to be used by the student to supplement the instruction obtained from the traditional lecture and demonstration method, a class text entitled Calculus and Analytic Geometry by Stein (49) was used. The material in the calculus course called Mathematics 121 is designed to be studied in a period of one quarter. The contents of a class text were then divided into twenty suitable CAI units. Each unit contains an explanation of the concepts involved, at least one solved problem and twenty-five unsolved problems.

The explanation of the concepts involved in each unit is aimed at providing the student with the tools to work. The solved problem is a detailed illustration of what the author expects of the student in the problems to be solved later. In the three to five partially solved problems, the student completes the blanks shown. With this method, the student learns each step that is required to achieve the desired results. If a wrong answer is filled in the blank, there will be a "wrong" response from the computer. The student can then erase the wrong answer and insert the right answer, go back to previous information to be acquainted with the material, strike a HELP key or ask the instructor for help. If a right answer is filled in the blank, there is a "right" response from the computer and the student can continue learning.

At the end of each unit there are twenty-five multiple-choice problems. The probability of getting one problem right out of five problems by guessing is 0.2. These twenty-five problems encompass all the subject matter in each unit. Therefore, the successful completion of these problems enhances the student's understanding of the material.

The final step in the development of the calculus units involved editing and revising the curriculum material. The first CAI unit was programmed on PLATO IV and students remarked that it facilitated their learning of calculus. Hopefully, with the development of computers, that will teach a wide variety of subject matter in an effective manner, the social ills brought on by an unequal distribution of quality education will be decreased.

Major recommendations resulting from this study include the following:

1. Before the use of PLATO or any CAI system, the student should be well-informed of the capabilities of the CAI system being utilized.
2. The CAI units in calculus developed in this study, will be tested in many classes.
3. Research in adapting the CAI system to other disciplines should be encouraged.

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X. APPENDIX A: CALCULUS CAI UNITS

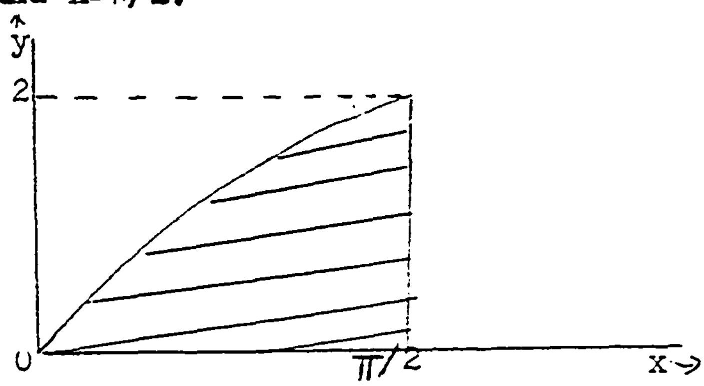
A. Unit 1

The First Fundamental Theorem of Calculus

If f is continuous on $[a,b]$ and if $f = d/dx(F)$ then $\int_a^b f(x)\,dx = F(b) - F(a)$.
F should be expressible in terms of polynomials, logarithms, exponentials, trigonometric functions, their inverses or any composition of these functions.

Solved problem:

Find the area bounded by $y = 2\sin x$ the x-axis and between $x = 0$ and $x = \pi/2$.



Solution:

$f(x) = 2\sin x$ is continuous on $[0, \pi/2]$

$f = d/dx(f)$ therefore, $2\sin x = d/dx(-2\cos x)$. $F(x) = -2\cos x$

Required area $\int_0^{\pi/2} 2\sin x\,dx$

$= 2\cos(\pi/2) + 2\cos 0 = -2(0-1) = 2$.

Now try solving the following problems:

1. $\int_0^1 e^{2t}\,dt$,

Solution:

$f(t) = e^{2t}$ is continuous on $[0,1]$

$f = d/dt(F)$, $e^{2t} = d/dt(\longrightarrow)$, $F(t) = e^{2t}/2$

$\int_0^1 e^{2t}\,dt = \longrightarrow$

therefore, $\int_0^1 e^{2t}\,dt = (e^2-1)/2$

2. $\int_1^2 (8/x^2)\,dx$

<u>Solution</u>:

$f(x)$ $8/x^2$ is continuous on $[1,2]$

$f = d/dx(F)$, $8/x^2 = d/dx(\longrightarrow)$, $F = -8/3x^3$

$\int_1^2 8x^{-2}\,dx = \longrightarrow$

therefore, $\int_1^2 8x^{-2}\,dx = -8(2)^{-3}/3+8(1)^{-3}/3 = 7/3$

3. $\int_2^3 x^3\,dx$

<u>Solution</u>:

$f(x)$ x^3 is continuous on $[\longrightarrow]$

f $d/dx(F)$, $x^3 = d/dx(\longrightarrow)$, $F = x^4/4$

$\int_2^3 x^3\,dx = \longrightarrow$

therefore, $\int_2^3 x^3\,dx = 3^4/4 - 2^4/4$ $65/4$.

4. $\int_1^e (7/x)\,dx$

<u>Solution</u>:

$f(x) = 7/x$ is continuous on $[1,e]$

$f = d/dx(F)$, $7/x = d/dx(\longrightarrow)$, $F = 7\ln x$

$\int_1^e (7/x)\,dx = \longrightarrow$

therefore, $\int_1^e (7/x)\,dx = 7\ln e - 7\ln 1 = 7$.

5. $\int_0^1 4x^3 \exp(x^4)\,dx$

<u>Solution</u>:

$f(x) = 4x^3 \exp(x^4)$ is continuous on $[\longrightarrow]$

$f = d/dx(F)$, $4x^3 \exp(x^4) = d/dx(\longrightarrow)$, $F = \exp(x^4)$

$\int_0^1 4x^3\exp(x^4)\,dx = e^1 - e^0 = e-1$

HELP1:

Remember the differentiation of the following functions:

1. $d/dx\,(\sin ax) = a \cos ax$, where a is a constant.
2. $d/dx\,(\cos ax) = -a\sin ax$, where a is a constant.
3. $d/dx(x^n) = nx^{n-1}$, where n is a constant.
4. $d/dx(e^{ax}) = ae^{ax}$, where a is a constant.

B. Unit 2

The Second Fundamental Theorem of Calculus

Let f be continuous on the interval $[a,b]$ let $y(x) = \int_a^x f(t)\,dt$ for $a < x \leq b$, then y is differentiable and its derivative is f.

$y^1(x) = f(x)$ or $d/dx(\int_0^x f(t)\,dt) = f(x)$

Trial problem:

Let $f(t) = t^3$ and $y(x) = \int_a^x t^3\,dt$

Find $y'(x)$.

Solution:

1. Method A

$y(x) = \int_a^x t^3\,dt \quad x^4/4 - a^4/4$

$y'(x) = d/dx(x^4/4 - a^4/4) = 4x^3/4 - 0 = x^3$

therefore, $y'(x) = x^3$.

2. Method B

$y(x) = \int_a^x t^3\,dt$

Let $u = x$, the upper limit of integration.

Then $du/dx = 1$.

Let $dy/du = u^3$, then $dy/du = x^3$ since $u = x$.

$y'(x) = dy/dx = dy/du.\ du/dx = x^3(1) = x^3$

Therefore, $y'(x) \quad x^3$

Since method 2, involves differentiation only, it is an easier method to solve these problems than method 1 which may involve functions which are difficult to integrate. Method 2 will therefore be used in solving the problems in this unit.

Now try solving the following problems:

1. If $y(x) = \int_2^x t^4\,dt$, find $y'(x)$.

<u>Solution</u>:

Let $u = \longrightarrow$, the upper limit of integration.

Then $du/dx = 1$

Also let $dy/du = u^4$

Substituting u for x, $dy/du = \longrightarrow$

$y'(x) = dy/dx = dy/du.\ du/dx = \longrightarrow$

therefore $y'(x) = 4x^3$.

2. If $y(x) = \int_1^{x^2} t^{1/2}\,dt$. Find $y'(x)$.

<u>Solution</u>:

Let $u = \longrightarrow$, the upper limit of integration.

Then $du/dx = 2x$.

Also let $dy/du = u^{1/2}$

Substituting u for x^2, $dy/du = \longrightarrow$

$y'(x) = dy/dx = dy/du.\ du/dx = \longrightarrow$

therefore, $y'(x) = 2x^2$

3. If $y(x) = \int_0^x \tan^2 t\,dt$, find $y'(x)$.

<u>Solution</u>:

Let $u = \longrightarrow$, the upper limit of integration

$du/dx = 1$

Also let $dy/du = \tan^2 u$

Substituting u for x, $dy/du = \longrightarrow$

$y'(x) = dy/dx = dy/du.\ du/dx = \longrightarrow$

$\therefore \quad y'(x) = \tan^2 x$

4. If $y(x) \int_1^{x^3} \sin 3t\,dt$, find $y'(x)$.

Solution:

Let $u = \longrightarrow$, the upper limit of integration.

Then $du/dx \quad 3x^2$

Also let $dy/du \quad \sin 3u$

Substituting u for x^3, $dy/du = \longrightarrow$

$y'(x) = dy/dx = dy/du.\ du/dx = \longrightarrow$

Therefore, $y'(x) = 3x^2 \sin 3x^3$

5. If $y(x) \int_0^{\sqrt{x}} \sqrt{1-t^3}\,dt$, find $y'(x)$.

Solution:

Let $u = \longrightarrow$, the upper limit of integration.

Then $du/dx = 1/2x^{-1/2}$

Also let $dy/du = \sqrt{1-u^3}$

Substituting u for $x^{1/2}$, $dy/du = \longrightarrow$

$y'(x) = dy/dx = dy/du.\ du/dx = \longrightarrow$

Therefore $y'(x) = (\quad (1-x3/2)/x)/2$.

C. Unit 3

Computing Integrals of by Using Formulas

Up to this unit, the method of evaluating an integral has been by the fundamental theorems of calculus. Now, eleven formulas will be used to evaluate integrals of some common functions.

1. $\int (f+g)\, dx = \int f dx + \int g\, dx$

   where f and g are functions of x.

2. $\int kf dx = k \int f dx$

   where k is any constant and f is a function of x.

3. $\int x^n\, dx = x^{n+1}/(n+1) + c$

   where n and c are constants and $n \neq -1$.

4. $\int \sin a\, x\, dx = -a \cos ax + C$

   where a and C are constants

5. $\int \cos ax\, dx = a \sin ax + C$

   where a and C are constants

6. $\int \tan x\, dx = \ln|\sec x| + C$

   where C is a constant

7. $\int \cot x\, dx = \ln|\sin x| + C$

8. $\int \sec x\, dx = \ln|\sec x + \tan x| + C$

   where C is a constant

9. $\int \csc x\, dx = \ln|\csc x - \cot x| + C$

   where C is a constant

10. $\int \sec^2 x\, dx = \tan x + C$

where C is a constant

11. $\int \csc^2 x \, dx = -\cot x + C$

where C is a constant

A quick method, of knowing whether the integral of a function is correct or not, is by differentiating the obtained function. If the original function is obtained, then the integral of the function is correct.

e.g. From formula 11 above, differentiate -cot x.

i.e. $d/dx(-\cot x) = d/dx(-\cos x)/\sin x$

$= (\sin x \, d/dx(-\cos x) - (-\cos x) \, d/dx) / \sin^2 x$

$= (\sin x \sin x + \cos x \cos x)/\sin^2 x$

$= (\sin^2 x + \cos^2 x)/\sin^2 x = 1/\sin^2 x = \csc^2 x$

therefore, $\int \csc^2 x \, dx = -\cot x + C.$

It is advantageous to be able to choose which formula or combination of formulas to be used.

Solved problems

1. $\int x^3 \, dx$

For this problem, formula 3 is the most appropriate. Substituting 3 for n, the result becomes

$\int x^3 \, dx = x^{3+1} \; (3+1) + C$

therefore, $\int x^3 \, dx = x/4^4 + C$

2. $\int 9 \, x^4 \, dx$

In this problem, formula 2 and 3 are the combination of formulas to use. k is represented by 9 and f is represented by x^4 in formula 2. Then Formula 3 is used next.

i.e. $\int 9\ x^4\ dx = 9\int x^4\ dx = 9\ x^{4+1}/(4+1) + C$

$= 9\ x^5/5 + C$

therefore, $\int 9\ x^4\ dx = 9x^5/5 + C$

Now try solving the following problems:

1. $\int x^9\ dx$

Apply formula 3 to this problem.

$\int x^9\ dx = \longrightarrow + C$

$\therefore \quad \int x^9\ dx = x^4/4.$

2. $\int 7\sin 4x\ dx$

Apply formula 2 and formula 4 to this problem and replace k by 7 and a by 4.

$\int 7e^{3x}\ dx = \longrightarrow e^{3x} + C$

therefore, $\int 7e^{3x}\ dx = (7/3)e^{3x} + C$

3. $\int (x^4 + 3x^3 + 6)\ dx$

Apply formula 1 and formula 3 to this problem.

$\int (x^4 + 3x^3 + 6)\ dx = \longrightarrow + C$

therefore, $\int (x^4 + 3x^3 + 6)\ dx = x^5/5 + 3x^4/4 + 6x + C$

4. $\int \tan 4x\ dx$

Apply formula 6 to this problem. The difference between this problem and formula 6 is that there is a figure 4 in this problem which is nonexistent in formula 6. Differentiate 4x with respect to x and the result is 4. Now rewrite the integral to be

$\int \tan 4x \, dx = \longrightarrow \int \tan 4x. \; 4 \, dx.$

The constant 1/4 is needed to make the right hand side of the equation equal to the left-hand side of the equation.

therefore, tan 4x dx (1/4) tan 4x. 4 dx

Apply formula 6 and the result becomes

$(1/4) \int \tan 4x. \; 4dx = \longrightarrow + C$

therefore, $\int \tan 4x \, dx = (1/4) \ln |\sec 4x| + C.$

5. $\int \sec^2 t \, dt$

This problem is the same as that shown in formula 10, except that in this problem the variable t replaces the variable x

$\int \sec^2 t \, dt = \longrightarrow + C$

therefore, $\int \sec^2 t \, dt = \tan t + C$

D. Unit 4

Computing Integrals of the Type

$\int f'(x)/f(x)\,dx$, $\int a^x\,dx$ and $\int e^{ax}\,dx$

Having studied differentiation and elementary integration, it is noticeable that the integration formulas arrived at, are obtained from standard differentiation formulas.

e.g. $d/dx\,(3x^7) = 21x^6 = f'(x)$

therefore, $\int f'(x)\,dx = \int 21x^6 dx = 3x^7 = f(x)$

Solved problem:

Find $\int 7x/(3+x^2)\,dx$

Solution:

Let $I = \int 7x/(3+x^2)\,dx$

Differentiating the denominator, $f(x) = 3+x^2$

therefore, $f'(x) = 2x$, $f'(x)/f(x) = 2x/(3+x^2)$

Multiply I by an appropriate constant so that the integral is equal to I.

therefore, $I = (7/2)\int 2x/(3+x^2)\,dx = (7/2)\ln|3+x^2| + C$

where C is a constant of integration.

Formulas

1. $\int f'(x)/f(x)\,dx = \ln|f(x)| + C$

   where C is a constant

2. $\int a^x dx = a^x/\ln a + C$, $a > 0$, $a \neq 1$

   where a and C are constants

3. $\int e^{ax}\,dx = e^{ax}/a + C$

where a and C are constants

Now try solving the following problems

1. $\int x/(x^2+4)\,dx$

The denominator in the integral is $f(x) = \longrightarrow$

therefore, $f'(x) = \longrightarrow$, $f'(x)/f(x) = \longrightarrow$

Now multiply this integral by an appropriate constant so that the integral below is equal to the integral above.

$$\int x/(x^2+4)\,dx = \longrightarrow \int 2x/(x^2+4)\,dx$$

The answer is $\longrightarrow + C$

$$\int x/(x^2+4)\,dx \quad (1/2)\ln\left|x^2+4\right| + C$$

2. $\int e^{4x}\,dx$

This is a direct application of theorem 3 and replacing a by 4.

$$\int e^{4x}\,dx = \longrightarrow + C$$

$$\int e^{4x}\,dx = e^{4x}/4 + C.$$

3. $\int 7^{5x}\,dx$

Differentiate 5x with respect to x and rewrite the integral so that formula 2 can be used.

$$\int 7^{5x}\,dx = (1/5)\int 7^{5x}(5dx) = \longrightarrow + C$$

$$\int 7^{5x}\,dx = (1/5)7^{5x}/\ln 7 + C$$

4. $\int x+3/(x+4)\ dx$

Divide $x+3$ by $x+4$

$x+3/(x+4)=1 - \longrightarrow$

i.e. $\int(x+3)/(x+4)\ dx = \longrightarrow\ + C$

therefore, $\int(x+3)/(x+4)\ dx = x - \ln(x\ \ 4) + C$

5. $\int \sin 2x/(1+\cos 2x)\ dx$

The denominator in the integral is $f(x) = \longrightarrow$

$f'(x) = \longrightarrow$, $f'(x)/f(x) = \longrightarrow$

Now multiply this integral by an appropriate constant so that the integral below is equal to the integral above.

$\int \sin 2x/(1+\cos 2x)\ dx = \longrightarrow$

$= \longrightarrow \int -2\sin 2x(1+\cos 2x)\ dx$

The answer is $\longrightarrow\ + C$

therefore, $\int \sin 2x/(1+\cos 2x)\ dx =$

$-(1/2)\ln|1+\cos 2x| + C$

E. Unit 5

Integration by Trigonometric Substitutions

Trigonometric substitutions are usually used for integrands which contain any of these three forms

$\sqrt{a^2-b^2x^2}$, $\sqrt{b^2x^2-a^2}$, and $\sqrt{a^2+b^2x^2}$

First form of substitution:

For $\sqrt{a^2-b^2x^2}$, use $x=(a/b)\sin t$, $dx=(a/b)\cos t\,dt$

therefore, $\sqrt{a^2-b^2x^2}=\sqrt{a^2-b^2((a^2/b^2)\sin^2 t)}=\sqrt{a^2-a^2\sin^2 t}$

$a\sqrt{1-\sin^2 t}=a\cos t.$

Second form of substitution:

For $\sqrt{b^2x^2-a^2}$, use $x=(a/b)\sec t$, $dt=(a/b)\sec t\tan t$

therefore, $\sqrt{b^2x^2-a^2}=\sqrt{b^2((a^2/b^2)\sec^2 t)-a^2}=\sqrt{a^2\sec^2 t-a^2}$

$=a\sqrt{\sec^2 t-1}=a\tan t$

Third form of substitution

For $\sqrt{a^2+b^2x^2}$, use $x=(a/b)\tan t$, $dt=(a/b)\sec^2 t\,dt$

$a\sqrt{1+\tan^2 t}=a\sec t$

Some useful formulas:

1. $\int(a^2-x^2)^{-1/2}dx=\arcsin(x/a)+C$

2. $\int(a^2+x^2)^{-1}dx=(1/a)\arctan(x/a)+C$

3. $\int 1/x\sqrt{x^2-a^2}\,dx = \text{arc sec }(x/a)+C$

4. $\int (x^2-a^2)^{-1}\,dx = (1/2a)\ln|(x-a)/(x+a)| + C$

5. $\int 1/(a^2-x^2)\,dx = (1/2a)\ln|(a+x)/(a-x)|+C$

6. $\int 1/\sqrt{x^2+a^2}\,dx = \ln(x+\sqrt{x^2+a^2})+C$

7. $\int 1/\sqrt{x^2-a^2}\,dx = \ln|x + x^2-a^2|+C$

8. $\int\sqrt{a^2-x^2}\,dx = (1/2)x\sqrt{a^2-x^2}+(a^2/2)\text{arc sin }(x/a)+C$

9. $\int\sqrt{a^2+x^2}\,dx = (1/2)x\sqrt{a^2+x^2}+(a^2/2)\ln(x+\sqrt{x^2+a^2})+C$

10. $\int\sqrt{x^2-a^2}\,dx = (1/2)x\sqrt{x^2-a^2} - (a^2/2)\ln|x+\sqrt{x^2-a^2}|+C$

Now try solving the following problems:

1. $\int(\sqrt{81-16x^2})/x\,dx$

<u>Solution:</u>

Use the first form of substitution.

$x = 9/4 \sin t,\ dx = \longrightarrow$

$\sqrt{81-16x^2} = \longrightarrow$

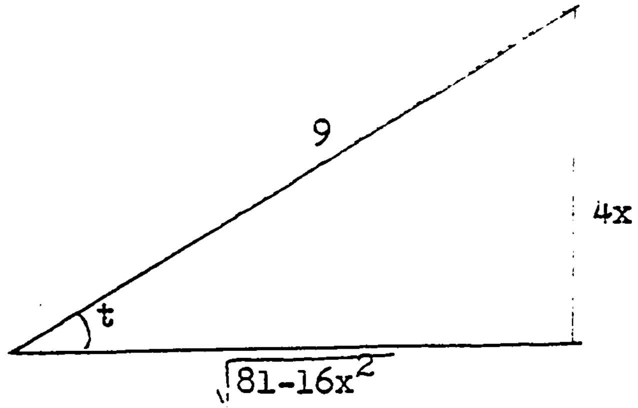
$\sqrt{81-16x^2} = 9\cos t$

$\int (\sqrt{81-16x^2})/x\,dx = \int 9\cos t/((9/4)\sin t\,dt)$

$= 9\int (1-\sin^2 t)/\sin t = 9\int(\csc t-\sin t)\,dt = \longrightarrow$

$= (\sqrt{81-16x^2})/x\,dx = 9\ln|\csc t-\cot t| + 3\cos t + C$

Substitute $x = (9/4)\sin t$. The right angled triangle below will be of some use.



$\int \sqrt{81-16x^2}/d\ dx \ = \longrightarrow$

therefore, $\int \sqrt{81-16x^2}\ /x dx = 9\ln \left| 9/4x - (\ \ 81-16x^2)/4x \right|$
$+ (\sqrt{81-16x^2})/3$

2. $\int x^2/\sqrt{x^2-16}\ dx$

Use the second form of substitution

$x = 4\ \sec t;\ dx = \longrightarrow$

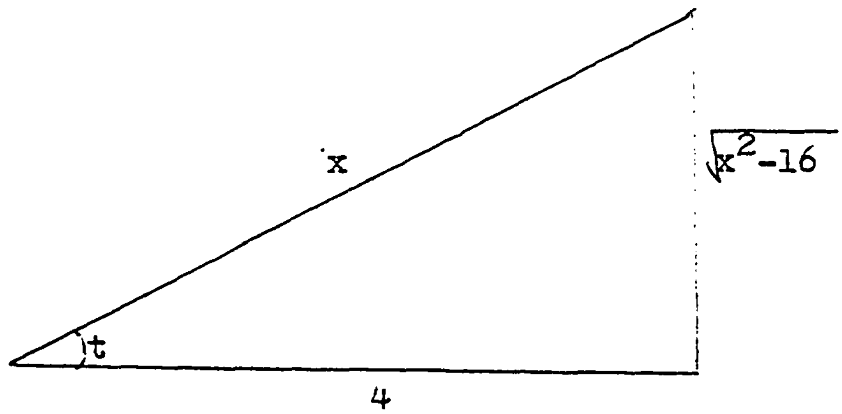
$\sqrt{x^2-16} = 4\ \tan t$

$\int x^2/\sqrt{x^2-16}\ dx = \int 16\sec^2 t(4\sec t\ \tan t)/4\ \tan t\ dt$

$\int 16\sec^3 t\ dt = 16 \int \sec^3 t\ dt \ = \ \longrightarrow$

$\int x^2/\sqrt{x^2-16}\ dx \ \ 8\sec t\ \tan t + 8\ln \left|\sec t + \tan t\right| + C$

Substitute $x = 4\sec t$ the right angle triangle below will be of some use.



$\int x^2/\sqrt{x^2-16}\,dx = \longrightarrow$

therefore, $x^2/\sqrt{x^2-16}\,dx$

$= x\sqrt{x^2-16}/2 + 8\ln|x/4 + \sqrt{x^2-16}/4| + c$

3. $\int 7/(x^2\sqrt{9+x^2})\,dx$

<u>Solution</u>:

Use the third form of substitution

$x = 3\tan t,\ dx = \longrightarrow$

$\sqrt{9+x^2} = 3\sec t$

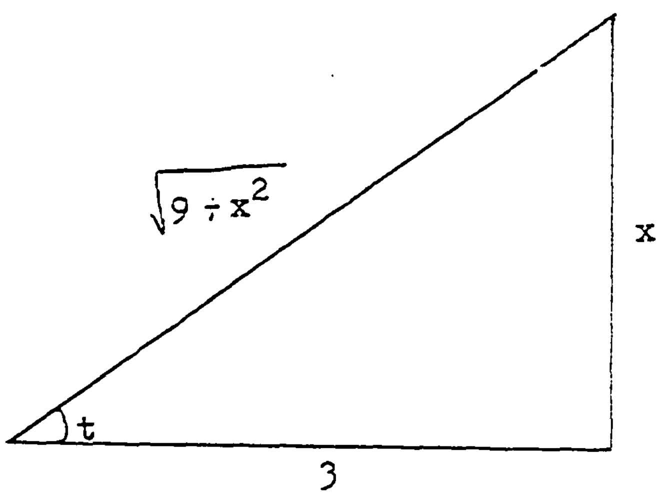
$\int 7/(x^2\sqrt{9+x^2})dx = \int 21\sec^2 t/9\tan^2 t(3\sec t)dx$

$= 7\int \sec t/9\tan^2 t\,dt$

$= (7/9)\int \sin^{-2} t\cos t\,dt = \longrightarrow$

$\int 7/(x^2\sqrt{9+x^2})\ dx = -(7/9)/\sin t + C$

Substitute $x = 3\tan t$. The right angle below will be of some use.



$\int 7/(x^2\sqrt{9+x^2})\ dx = \longrightarrow$

therefore, $\int 7/(x^2\sqrt{9+x^2})\ dx = -7\sqrt{9+x^2}/9x + C$

F. Unit 6

Integration by Parts

Theorem:

If u and v are differentiable functions of x

$$d(uv) = udv + vdu$$

$$\text{i.e. } udv \quad d(uv) - vdu$$

$$\text{Integrating, } \int udv = uv - \int vdu$$

To use the method of integration by parts, the given integral must be separable into two parts, u and dv. u should be chosen so that it is easily differentiable and dv must be chosen so that it is easily integrable.

Solved problem:

Find $\int xe^x \, dx$

Solution:

x is easily differentiable and e^x dx is easily integrable.

Let $u = x$ and $dv = e^x \, dx$

therefore, $du = dx$ and $v = e^x$

From the theorem of integration by parts,

$\int x\, e^x dx = xe^x - \int e^x dx = xe^x - e^x$

therefore, $\int xe^x dx = e^x(x-1) + C$

Now try solving the following problems

1. Evaluate $\int 12x\cos x \, dx$

x is easily differentiable and 12 cosx dx is easily integrable.

Let $u = \rightarrow$ and $dv = \rightarrow$

$du = \rightarrow$, $v = \rightarrow$

From the theorem of integration by parts,

therefore, $\int 12x\cos x \, dx = \rightarrow - \int \rightarrow$

therefore, $\int 12x\cos x \, dx = 12x\sin x + 12\cos x + C$

2. Evaluate $3\int \arcsin x \, dx$

3arc sinx is easily differentiable and dx is easily integrable.

Let $u = \rightarrow$ and $dv = \rightarrow$

$du = \rightarrow$, $v = \rightarrow$

From the theorem of integration by parts,

$3\int \arcsin x \, dx = \rightarrow - \int \rightarrow$

$3 \quad \arcsin x \, dx \quad 3x\arcsin x \quad - \int 3x/\sqrt{/-x^2} \, dx$

To evaluate the integral on the right,

substitute $t^2 = 1-x^2$, then $2tdt = -2xdx \cdot tdt = \rightarrow$

$- \int 3x/\sqrt{1-x^2} \quad - \int -3t/t \, dt = \int 3dt = 3t+C = 3\sqrt{1-x^2}+C$

therefore, $\int 3\arcsin x \, dx = \rightarrow$

$\int 3\arcsin x \, dx = 3x\arcsin x + 3\sqrt{1-x^2}+C$

3. Evaluate $9\int x^2 \ln x \, dx$

9lnx is easily differentiable and x^2dx is easily integrable.

Let u = $\longrightarrow$ and dv = $\longrightarrow$

therefore, du = $\longrightarrow$, v = $\longrightarrow$

From the theorem of integration by parts,

$9\int x^2 \ln x \, dx = \longrightarrow - \int \longrightarrow$

$9\int x^2 \ln x = 3x^3 \ln x - 3\int x^2 dx$

$= 3x^3 \ln x - \longrightarrow$

therefore, $9\int x^2 \ln x \, dx = 3x^3 \ln x - x^3 + C$

4. Evaluate $2\int e^x \cos x \, dx$

Let $I = 2\int e^x \cos x \, dx$

$2e^x$ is easily differentiable and cosx dx is easily integrable.

Let u = $\longrightarrow$ and dv = $\longrightarrow$

therefore, du = $\longrightarrow$, v = $\longrightarrow$

From the theorem of integration by parts,

$I = \longrightarrow - \int \longrightarrow$

$I \quad 2e^x \sin x - 2\int e^x \sin x \, dx$.................(1)

In the integral $2\int e^x \sin x \, dx$, $2e^x$ is easily differentiable and sinx dx is easily integrable.

Let u = $\longrightarrow$ and dv = $\longrightarrow$

therefore, du = $\longrightarrow$, v = $\longrightarrow$

From the theorem of integration by parts,

$2\int e^x \sin x \, dx = - \int \longrightarrow$

$2\int e^x \sin x \, dx = -2e^x \cos x + \int 2 e^x \cos x \, dx$

i.e. $2\int e^{x} \sin x = -2e^{x} \cos x + I$ ....(2)

Substitute equation (2) in equation (1)

$I = \longrightarrow$

$I = 2e^{x} \sin x + 2e^{x} \cos x - I$

$2I = 2e^{x} \sin x + 2e^{x} \cos x$

$I = 2(e^{x} \sin x + e^{x} \cos x)/2 \quad e^{x} \sin x + e^{x} \cos x + C$

therefore, $2\int e^{x} \cos x \, dx = e^{x}(\sin x + \cos x) + C$.

5. Evaluate $\int 15x\sqrt{1+x}\, dx$

Let $I = 15\int x\sqrt{1+x}\, dx$

$15x$ is easily differentiable and $\sqrt{1+x}$ is easily integrable.

Let $u = \longrightarrow$ and $dv = \longrightarrow$

$du = \longrightarrow$, $v = \longrightarrow$

From the theorem of integration by parts

$I = \longrightarrow \quad - \int \longrightarrow$

$I = 10x(1+x)^{3/2} - 10\int (1+x)^{3/2}\, dx$

Integrate $10\int (1+x)^{3/2}\, dx$.

$I = 10x\,(1+x)^{3/2} - 4(1+x)^{5/2} + C$

therefore, $\int 15x\sqrt{1+x}\, dx = 10x(1+x)^{3/2} - 4(1+x)^{5/2} + C$

G. Unit 7

Completing the Square and Integration by Completing the Square.

To complete the square of the function $f(x)=ax^2+bx+c$, rewrite $f(x)$ as

$$ax^2+bx+c = a(x+(b/2a))^2+(c-(b^2/4a))$$

Reduce the right hand-side of the equation.

$$a(x+(b/2a))^2+(c-(b^2/4a))=a(x^2+2bx/2a+b^2/4a^2)+c-b^2/4a$$

$$= ax^2+bx+(b^2/4a)+c-(b^2/4a)$$

$$= ax^2+bx+c$$

$ax^2+bx+c = a(x+b/2)^2+(c-b^2/4a)$ is an identity.

Solved problem

Complete the square of $6x^2+2x+9$ and integrate $\int 1/(6x^2+2x+9)\ dx$

Solution:

In this problem, substitute a for 6, b for 2 and c for 9, in the identity above.

$$6x^2+2x+9 = 6(x+2/2(6))^2+(9-4/4(6))$$

$$= 6(x+(1/6))^2+(9-(1/6))$$

$$= 6(x+(1/6))^2+53/6$$

therefore, $6x^2+2x+9 = 6(x+(1/6))^2+53/6$.

$$\int 1/(6^2+2x+9)\ dx = \int 1/(6(x+(1/6))^2+53/6\ )$$

Use the method of substitution used in Unit 5.

Let $u=\sqrt{6}(x+(1/6))$ then $du=\sqrt{6}\ dx$

The integral becomes $1/\sqrt{6}\int 1/(u^2+53/6)$

From the formulas in Unit 5

$(1/\sqrt{6})\int 1/(u^2+53/6)\ dx = (1/\sqrt{6}).(\sqrt{6}/\sqrt{53})$ arc $\tan\sqrt{6}u/\sqrt{53}+C$

$=(1/\sqrt{53})$ arc $\tan\sqrt{6}\sqrt{6}(x+1/6)/\sqrt{53}+C$

$=(1/\sqrt{53})$ arc tan $6\ (x+1/6)/\sqrt{53}+C$

therefore, $\int 1/(6x^2+2x+9)\ dx$

$=(1/\ 53)$ arc tan $6\ (x+1/6)/\sqrt{53}+C$

Now try solving the following problems.

1. Find $\int 1/(x^2+2x+2)\ dx$

<u>Solution</u>:

First complete the square of the denominator.

From the identity for completing squares

$x^2+2x+2 = \longrightarrow$

$x^2+2x+2=(x+1)^2+1$

$\int(x^2+2x+2)^{-1}\ dx = \int((x+1)^2+1)^{-1}\ dx$

Let $u=x+1$, $du = \longrightarrow$

$\int 1/((x+1)+1)\ dx = \int 1/(u^2+1)\ dx$

Use the trigonometric substition $u=\tan t$ as shown in Unit 5.

$\int(u^2+1)^{-1}du = \longrightarrow$

$\int(u^2+1)^{-1}du = \arctan u =$ arc $\tan(x+1)+C$

therefore, $\int(x^2+2x+2)^{-1}dx =$ arc $\tan(x+1)+C$.

2. Evaluate $\int 1/(3x^2+6x+1)\ dx$

<u>Solution</u>:

First complete the square of the denominator. From the identity of completing squares,

$3x^2+6x+1 = \longrightarrow$

$3x^2+6x+1 = 3(x+1)^2+1/4$

Let $u = \sqrt{3}(x+1)$, $du = \longrightarrow$

$\int 1/(3x^2+6x+1)\,dx = 1/\sqrt{3}\int 1/(u^2+1)du$

Use the trigonometric substitution $u = \tan t$ as shown in Unit 5.

$(1/\sqrt{3})\int (u^2+1)^{-1}du = \longrightarrow$

$(1/\sqrt{3})\int 1/u^2+1)\,du = (1/\sqrt{3})\ \text{arc}\tan u$

$= (1/\sqrt{3}\ \text{arc}\tan\sqrt{3}(x+1)+C$.

3. Find $\int 1/(4x^2+2x-3)\,dx$

<u>Solution</u>:

First complete the square of the denominator. From the identity of completing squares,

$4x^2+2x-3 = \longrightarrow$

$4x^2+2x-3 = 4(x+(1/4))^2 - 13/4$

Let $u = 2(x+1/4)$, $du = \longrightarrow$

$\int 1/\sqrt{4x^2+2x-3}\,dx = (1/2)\int 1/\sqrt{u^2-13/4}\,du$

Use formula 7 in Unit 5.

$1/2\sqrt{u^2-13/4}\,du = \longrightarrow$

$$\int 1/2\sqrt{u^2-13/4}\ du = (1/2)\ln\left|u+ u^2-a^2\right|+C$$

$$=(1/2)\ln\left|2(x+1/4)+\sqrt{4(x+1/4)^2-13/4}\right|+C$$

therefore, $\int (4x^2+2x-3)\ dx$

$$(1/2)\ln\left|2(x+1/4)+\sqrt{4(x+1/4)^2-13/4}+C.\right.$$

H. Unit 8

Integration by Partial Fractions

A polynomial is a function of the form $a_0+a_1x+a_2x^2+\ldots+a_nx^n$. A rational function is a function of the form $f(x)/g(x)$ where $f(x)$ and $g(x)$ are polynomials. $f(x)/g(x)$ is called proper if the degree of $f(x)$ is less than the degree of $g(x)$. $f(x)/g(x)$ is called improper if the degree of $f(x)$ is greater than the degree of $g(x)$.

To reduce $f(x)/g(x)$ where $f(x)$ and $g(x)$ are polynomials, as the sum of partial fractions, the following cases should be considered:

Case 1. If the degree of $f(x)$ is greater than or equal to the degree of $g(x)$, divide $g(x)$ into $f(x)$ to obtain a quotient $g(x)$ and a remainder $r(x)$ Then $f(x)/g(x)=q(x)+r(x)/q(x)$.

Case 2. If the degree of $f(x)$ is less than the degree of $g(x)$, then factorize $g(x)$ to factors that are irreducible. In case $ax+b$ is a linear factor occuring n times in $g(x)$, then there corresponds a sum of n partial fractions of the form

$$f(x)/g(x)=f(x)/(ax+b)^n$$

$$=A_1/(ax+b)+A_2/(ax+b)^2+\ldots+A_n/(ax+b)^n$$

where a,b, A's and B's are constants.

In case $g(x)=ax^2+bx+c$ and $f(x)/g(x)$ is a proper function then

$f(x)/g(x)=(A_1x+B_1)/(ax^2+bx+c)$

where a,b,c,A_1 and B_1 are constants

If ax^2+bx+c appears n times in the factorization of $g(x)$, then

$$f(x)/g(x)\ f(x)/(ax^2+bx+c)^n$$
$$=(A_1x+B_1)/(ax^2+bx+c)+(A_2x+B_2)/(ax^2+bx+c)^2$$
$$..+(A_nx+B_n)/(ax^2+bx+c)^n$$

where a,b,c,A's and B's are constants.

<u>Solved problem</u>:

Decompose $(7x-1)/(x-1)(x+2)$ into partial fractions.

<u>Solution</u>:

Write $(7x-1)/\ (x-1)(x+2)=A/(x-1)+B/(x+2)$

where A and B are constants to be determined.

Multiply through by $(x-1)(x+2)$.

$7x-1=A(x+2)+B(x-1)$..............(1)

Let $x=-2$ in equation 1.

$7(-2)+1=B(-3$

$-15=B(-3)$

$B=5$

Let $x=1$ in equation 1.

$7(1)-1=A(3)$

$6 = A(3)$

$A = 2$

Therefore,

$(7x-1)/(x-1)(x+2) = 2/(x-1)+5/(x+2)$

Now try solving the following problems.

1. Express $(3x+1)/(x^2+2x-3)$ as a sum of partial fractions

<u>Solution</u>:

Write $(3x+1)/(x^2+2x-3) = (3x+1)/(x+3)(x-1) = \longrightarrow + \longrightarrow$

$(3x+1)/(x+3)(x-1) = A/(x+3) + B/(x-1)$

where A and B are constants to be determined.

Multiply through by $(x+3)(x-1)$.

$3x+1 = \longrightarrow + \longrightarrow$

$3x+1 = A(x-1)+B(x+3)$ ....(2)

Let $x=1$ in equation 2.

$3(1)+1 = A(1-1)\ B(1+3)$

$B = \longrightarrow$

$B = 1$

Let $x=-3$ in equation 2.

$3(-3)+1 = A(-3-1)+B(-3\ 3)$

$A = \longrightarrow$

$A = 2$

$3x+1/(x^2+2x-3) = \longrightarrow + \longrightarrow$

therefore, $(3x+1)/(x^2+2x-3) = 2/(x+3)+1/(x-1)$.

2. Decompose $(6x+4)/(9-x^2)$ into partial fractions.

<u>Solution</u>:

Write $(6x+4)/(9-x^2)=(6x-4)/(3-x)(3+x)= \longrightarrow + \longrightarrow$

$(6x+4)/(9-x^2) = A/(3-x) + B/(3+x)$

where A and B are constants to be determined.

Multiply through by $(3-x)(3+x)$.

$6x+4 = \longrightarrow + \longrightarrow$

$6x+4=A(3+x)+B(3-x)$ ....(3)

Let $x=-3$ in equation 3.

$6(-3)+4=A(3-3)+B(3+3)$

$B = \longrightarrow$

$B = -7/3$

Let $x=3$ in equation 3.

$6(3)+4=A(3+3)+B(3-3)$

$A = \longrightarrow$

$A = 11/3$

$(6x+4)/(9-x^2) = 11/(3(3-x)) \quad -7/(3(3-x))$

3. Express $(2x^2+7)/(2x^2+18x+28)$ as a sum of partial fractions.

<u>Solution</u>:

Since the degree of the numerator is equal to the degree of the denominator, divide the numerator by the denominator.

$$\begin{array}{r|l} & \quad 1 \\ \hline 2x^2\ 18x\ 28 & 2x^2+7 \\ & \underline{2x^2+18x+28} \\ & \quad -18x-21 \end{array}$$

$(2x^2+7)/(2x^2+18x+28 = 1 + (-18x-21)/(2x^2+18x+28)$

Write

$(-18x-21)/(2x^2+18x+28) = -3\ (6x+7)/(2(2x^2+18x+28))$

Therefore, $(2x^2+7)/(2x^2+18x+28)$

$= 1-3(6x+7)/(2(x^2+9x+28)$........................(4)

Write

$(6x+7)/(x^2+9x+28) = (6x+7)/(x+2)(x+7) = \longrightarrow + \longrightarrow$

$(6x+7)/(x+2)(x+7) = A/(x+2)+B/(x+7)$

where A and B are constants to be determined.

Multiply through by $(x+2)(x+7)$.

$6x+7 = \longrightarrow + \longrightarrow$

$6x+7 = A(x+7)+B(x+2)$.............................(5)

Let $x=-7$ in equation 5

$6(-7)+7 = A(-7+7) + B(-7+2)$

$-35 = B(-5)$

$B = \longrightarrow$

$B = 7$

Let $x=-2$ in equation 5

$6(-2)+7 = A(-2+7)+B(-2+2)$

$A = \longrightarrow$

$A = -1$

$(6x+7)/(x+2)(x+7) = \longrightarrow + \longrightarrow$

$(6x+7)/(x+2)(x+7) = -1/(x+2) + 7/(x+7)$..............(6)

Substitute equation 6 in equation 4.

$(2x^2+7)/(2x^2+18x+28) = \longrightarrow + \longrightarrow - \longrightarrow$

Therefore,

$(2x^2+7)/(2x^2+18x+28) = 1+3/(2(x+2)) - 21/(2(x+7))$

4. Evaluate $\int 8/(16-x^2)\,dx$

<u>Solution</u>

Write $8/(16-x^2) = 8/(4-x)(4+x)$

$8/(4-x)(4+x) = A/(4-x) + B/(4-x)$

where A and B are constants to be determined.

Multiply through by $(4-x)(4+x)$.

$8 = A(4+x) + B(4-x)$......................(7)

Let $x = -4$ in equation 7

$8 = A(4-4) + B(4+4)$

$B = \longrightarrow$

$B = 1$

Let $x = 4$ in equation 7

$8 = A(4+4) + B(4-4)$

$A = \longrightarrow$

$A = 1$

$8/(16-x^2) = \longrightarrow + \longrightarrow$

$8/(16-x^2) = 1/(4-x) + 1/(4+x)$

Integrating,

$\int 8/(16-x^2)\ dx = \int ((4-x)^{-1} + (4+x)^{-1})\ dx = \longrightarrow + C$

$\int 8/(16-x^2)\ dx = -\ln(4-x)\ \ln(4\ x) + C$

$= \ln(4+x)/4-x) + C$

$\int 8/(16-x^2)\ dx = \ln(4+x)/(4-x) + C$

I. Unit 9

Integration of Powers of Trigonometric Functions

The following identities are useful in integration of Powers of Trigonometric Functions.

1. $\sin^2 x + \cos^2 x = 1$
2. $1 + \tan^2 x = \sec^2 x$
3. $1 + \cot^2 x = \csc^2 x$
4. $\cos 2x = \cos^2 x - \sin^2 x$
5. $\cos 2x = 2\cos^2 x - 1$
6. $\cos 2x = 1 - 2\sin^2 x$
7. $\sin 2x = 2\sin x \cos x$
8. $2\sin x \cos y = \sin(x-y) + \sin(x+y)$
9. $2\sin x \sin y = \cos(x-y) - \cos(x+y)$
10. $2\cos x \cos y = \cos(x-y) + \cos(x+y)$

<u>Solved problem</u>

Evaluate $\int (\cos^2 x - \sin^2 x)\, dx$

<u>Solution</u>:

Use identity 4. $\cos^2 x - \sin^2 x = \cos 2x$

$$\int (\cos^2 x - \sin^2 x)\, dx = \int \cos 2x\, dx = (1/2)\sin 2x + C$$

Therefore,

$$\int (\cos^2 x - \sin^2 x)\, dx = (1/2)\sin 2x + C$$

Now try solving the following problems.

1. Evaluate $\int 72 \sin^3 3x \cos^5 3x\, dx$

<u>Solution</u>:

$$\sin^3 3x \cos^5 3x = \sin^2 3x \sin 3x \cos^5 3x$$

$$= (1-\cos^2 3x)\sin 3x \cos^5 3x$$

$$= \sin 3x \cos^5 3x - \sin 3x \cos^7 3x$$

Use the formula that $\int (f(x))^n f\ (x)\ dx = (f(x))^{n+1}/(n+1)$

Therefore, $\int 72 \sin^3 3x \cos^5 3x\ dx = \int \longrightarrow$

$$72 \int \sin^3 3x \cos^5 3x\ dx$$

$$= 72 \int (\cos^5 3x \sin 3x - \cos^7 3x \sin 3x)\ dx$$

Hence,

$$\int 72 \sin^3 3x \cos^5 3x\ dx = -4\cos^6 3x + 3\cos^8 3x + C.$$

2. Evaluate $\int 15 \csc^6 x\ dx$

<u>Solution</u>:

$$15\csc^6 x = 15\csc^2 x \csc^4 x = 15\csc^2 x(1+\cot^2 x)^2$$

$$= 15\csc^2 x(1+2\cot^2 x+\cot^4 x)$$

$$= 15\csc^2 x + 30 \cot^2 x\csc^2 x + 15 \cot^4 x\csc^2 x$$

Use the formula that states that

$$\int (f(x))^n f'(x)\ dx = (f(x))^{n+1}/(n+1) + C$$

Hence, $\int 15\csc^6 x\ dx$

$$\int (15\csc^2 x + 30\cot^2 x\csc^2 + 15 \cot^4 x \csc^2 x)\ dx$$

$$\int 15\csc^6 x\ dx = -15 \cot x - 10 \cot^3 x - 3 \cot^5 x + C.$$

3. Evaluate $\int 4\cot^3 2x\ dx$

Solution:

$$\cot^3 2x = \cot 2x \cot^2 2x = \cot 2x(\csc^2 2x-1)$$

$$=\cot 2x \csc^2 2x-\cot 2x$$

Use the formula that states that

$$\int (f(x))^n f'(x)\, dx = (f(x))^{n+1}/(n+1)+C$$

Hence,

$$\int 4\cot^3 2x\, dx = \int 4(\cot 2x + \cot^2 2x-\cot 2x)\, dx = \longrightarrow +C$$

$$\int 4\cot^3 2x\, dx = -\cot^2 2x + \ln|\csc 2x| + C$$

4. Evaluate $\int 3\tan^4 x\, dx$

Solution:

$$\tan^4 x = \tan^2 x \tan^2 x = \tan^2 x(\sec^2 x-1)$$

$$=\tan^2 x \sec^2 x-\tan^2 x$$

$$=\tan^2 x \sec^2 x-(1-\sec^2 x)$$

$$=\tan^2 x \sec^2 x-1 \sec^2 x$$

Use the formula that

$$(f(x))^n f'(x)\, dx = (f(x))^{n+1}/(n+1)+C$$

Therefore,

$$\int 3\tan^4 x\, dx = (3\tan^2 x \sec^2 x-3+3\sec^2 x)\, dx = \longrightarrow \quad +C$$

Hence,

$$\int 8\tan^4 x\, dx = \tan^3 x-3\tan x+3x+C$$

K. Unit 10

Integration by Miscellaneous Substitutions

Substitutions are often suggested by the form of the integrand. The following are good substitutions to use.

1. For an integrand of the form $(ax+b)^{1/n}$, use the substitution $ax+b=t^n$, where a,b are constants.
2. For an integrand of the form $\sqrt{a+bx+x^2}$, use the substitution $a+bx+x^2=(t-x)^n$, where a,b,n are constants.
3. For an integrand of the form $\sqrt{a+bx-x^2}=\sqrt{(c+x)(d-x)}$, use the substitution $a+bx-x^2=(c+x)^2t^2$ or the substitution $a+bx-x^2=(d+x)^2t^2$, where a,b,c,d are constants.
4. For an integrand involving trigonometric functions, use the substitution $\tan(x/2)=t$. On differentiating, $(1/2)\sec^2(x/2)\,dx=dt$, $\sec^2(x/2)\,dx=2dt$, $dx=2/\sec^2(x/2)$, $dt=2/(1+\tan^2(x/2))=2/(1+t^2)dt$

Also, $\sin x=2\tan(x/2)/(1+\tan^2(x/2))=2t/(1+t^2)$

$\cos x=(1-\tan^2(x/2))/(1+\tan^2(x/2))=(1-t^2)/(1+t^2)$

$\tan x=(2\tan(x/2))/(1-\tan^2(x/2))=2t/(1-t^2)$

Note that $\csc x=1/\sin x=(1+t^2)/2t$

$\sec x = 1/\cos x = (1+t^2)/(1-t^2)$

$\cot x = 1/\tan x = (1-t^2)/2t$

Solved problem: Evaluate $1/(1-\cos x)\,dx$

Let $\tan(x/2) = t$. From method 4 of substitution, $dx = 2/(1+t^2)\,dt$ and $\cos x = (1-t^2)/(1+t^2)$

$\int 1/(1-\cos x)\,dx = \int 2/(1+t^2)/(1-(1-t^2)/(1+t^2))\,dt$

$= \int 2/(2t^2)\,dt = \int 1/t^2\,dt$

$= -1/t + C = -1/\tan(x/2) + C$

Therefore

$\int 1/(1-\cos x)\,dx = -1/\tan(x/2) + C.$

Now try solving the following problems:

1. Evaluate $\int 1/(3+2\sin x)\,dx$

Solution:

Let $\tan(x/2) = t$, then $dx = \longrightarrow$ and $\sin x = \longrightarrow$

$\int 1/(3+2\sin x)\,dx = \longrightarrow$

$\int 1/(3+2\sin x)\,dx = \int (1/(1+t^2))/(3+4t/(1+t^2))\,dt$

$\int 1/(3t^2+4t+3)\,dt$

$= \int 1/(3((t+2/3)^2+5/9))\,dt = (1/\sqrt{5})\text{arc} \tan(t+2/3)/(\sqrt{5}/3) + C$

$= (1/\sqrt{5})\text{arc} \tan (3t+2)/\sqrt{5} + C$

Therefore

$\int 1/(3+2\sin x)\,dx = (1/\sqrt{5}) \tan (3\tan(x/2)+2)/\sqrt{5} + C$

2. Evaluate $\int \cos\sqrt{x}\,dx$

Solution:

Let $t = \sqrt{x}$, then $dt = \longrightarrow$

substitute t and dx in the integral.

$\int \cos\sqrt{x}\,dx = \longrightarrow$

$\int \cos\sqrt{x}\,dx = \int \cos t\ (2t\ dt) = 2\int t\cos t\ dt$

$= 2t\sin t - 2\int \sin t\ dt = 2t\sin t + 2\cos t + C$

$2\sqrt{x}\ \sin\sqrt{x} + 2\cos\sqrt{x} + C$

3. Find $\int 1/(x\sqrt{x^2+x+1})\ dx$

Solution:

Let $x^2+x+1 = (t-x)^2$.......................(1)

Solve for x in terms of t.

$x = \longrightarrow$, $dx = \longrightarrow$, $\sqrt{x^2+x+1} = \longrightarrow$

Solving for x in terms of t,

$x^2+x+1 = t^2-2xt+x^2$

$x(1+2t) = t^2-1$

$x = (t^2-1)/(1+2t)$

$dx = ((1+2t)(2t)-(t^2-1)(2))/(1+2t^2)\ dt$

$= (2t+4t^2-2t^2+2)(1+2t^2)\ dt$

$\int (2t^2+2t+2)/(1+2t)^2\ dt$

$\sqrt{x^2+x+1} = \sqrt{(t^2-1)^2/(1+2t)^2+(t^2-1)/(1+2t)+1}$

$$=\sqrt{((t^2-1)^2+(t^2-1)(1\ 2t)+(1+2t)^2)/(1+2t)}$$

$$=\sqrt{(t^4-2t^2+1+t^2+2t^3-2t-1+1+4t+4t^2)(1+2t)}$$

$$=\sqrt{(t^4+2t^3+3t^2+2t+1)/(1+2t)}=(t^2+t+1)/(1+2t)$$

$$\int 1/(x\sqrt{x^2+x+1})\ dx = \longrightarrow$$

$$\int 1/(x\sqrt{x^2+x+1})dx$$

$$=\int (2(t^2+t+1)(1+2t)(1+2t))/((1+2t)^2(t^2-1)(t^2+t+1))\ dt$$

$$=\int 2/(t^2-1)\ dt=\ln|(t-1)/(t\ 1)|+C$$

From equation 1.

$$x^2+x+1=(t-x)^2$$

Write t in terms of x.

$$t = \longrightarrow$$

$$x^2+x+1=(t-x)^2$$

$$\sqrt{x^2+x+1}=t-x$$

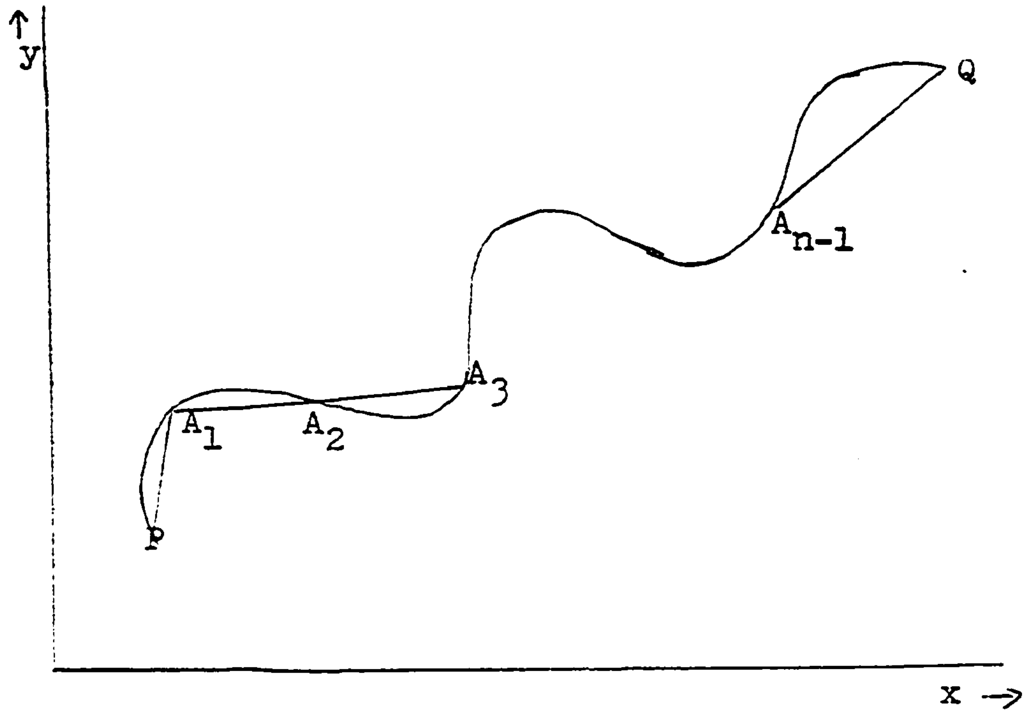
$$t=\sqrt{x^2+x+1}+x$$

$$\int 1/(x\sqrt{x^2+x+1})\ dx$$

$$=\ln|(\sqrt{x^2+x+1}+x-1)/(\sqrt{x^2+x+1}+x+1)|+C\ .$$

K. Unit 11

Computation of Cross-Sectional Length s



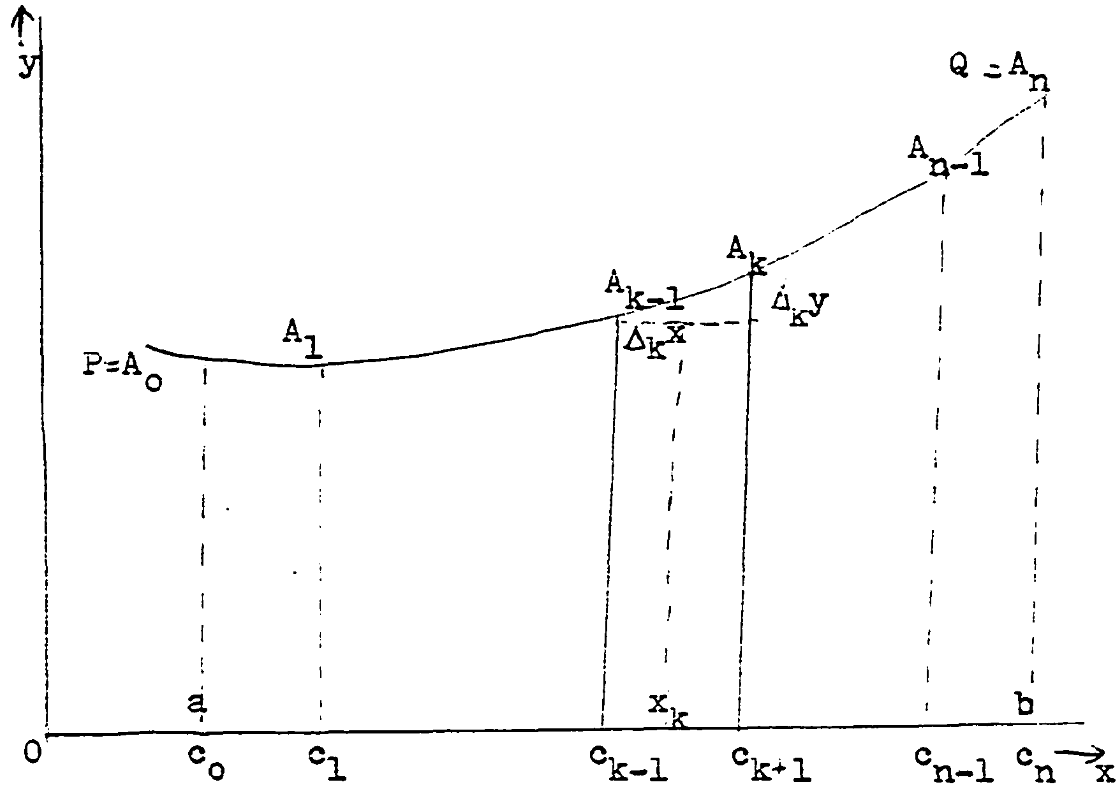
The length of an arc PQ is the limit of the sum of all the chords PA_1, $A_1A_2 \ldots\ldots\ldots A_{n-1}Q$. Let P(a,l) and Q(b,m) be two points on the curve $y = f(x)$ where $f(x)$ and $f'(x)$, its derivative are continuous on $a \leq x \leq b$, then the arc length s is given by $s = \int_{PQ} ds = \int_a^b \sqrt{1+(dy/dx)^2}\, dx$

Let P(a,l) and Q(b,m) be two points on the curve x h(y) where $h(y)$ and $h'(y)$, its derivative are continuous on $l \leq y \leq m$, then the arc length is given by $s = \int_{PQ} ds = \int_l^m \sqrt{1+(dx/dy)^2}\, dy$

Let $P(t=t_1)$ and $Q(t=t_2)$ be two points on the curve defined by $x=f(t)$, $y=g(t)$. Similarly, if the conditions of continuity are met, the length of PQ is

$$s = \int_{PQ} ds = \int_{t_1}^{t2} \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt$$

Derivation of the arc length formula



From the figure above, the interval $a \leq x \leq b$ is divided into points $c_0=a, c_1, c_2, \ldots, c_{n-1}, c_n=b$, with corresponding points $P=A_0, A_1, A_2, \ldots, A_{n-1}, A_n=Q$.

$$A_{k-1}A_k = \sqrt{(\Delta_k x)^2 \ (\Delta_k y)^2} = \sqrt{1+(\Delta_k y/\Delta_k x)^2}$$

There is at least one point $x=x_k$ on $A_{k-1}A_k$ such that $f'(x_k) = \Delta_k y/\Delta_k x$

$$A_{k-1}A_k = \sqrt{1+(f'(x_k))^2}, \quad c_{k-1} < x_k < x_k$$

Taking the limit as k tends to infinity.

$$PQ \lim_{n\to\infty} \sum_{k-1}^{n} \sqrt{1+(f'(x_k))^2} = \int_a^b \sqrt{1+(dy/dx)^2}\, dx$$

Solved problem

Find the length of the arc of $y \quad 2x^{3/2}/3$ from $x=1$ to $x=2$.

Solution:

$$y \quad 2x^{3/2}/3, \quad dy/dx \quad x^{1/2}, \quad 1+(dy/dx)^2 = 1+x$$

Arc length $s = \int_1^2 \sqrt{1+x}\, dx = \left[1/2\sqrt{1+x}\right]_1^2$

$= 1/2(1/\sqrt{3}+1) = (1+\sqrt{3})/(2\sqrt{3})$ units

Now try solving the following problems.

1. Find the length of the arc of the curve $x=2t^2$, $y=t^2$ from $t=0$ to $t=1$.

Solution:

$dx/dt = \longrightarrow \qquad , dy/dt = \longrightarrow$

$dx/dt = 4t \qquad , \qquad dy/dt = 2t$

$(dx/dt)^2 + (dy/dt)^2 = \quad \longrightarrow$

$(dx/dt)^2+(dy/dt)^2=(4t)^2+(2t)^2=16t^2+4t^2=20t^2$

The arc length $s=\int_0^1 \sqrt{20t^2}\,dt$

The arc length $s=\int_0^1 \sqrt{20t^2}\,dt=\sqrt{20}\int_0^1 t\,dt$

$=\sqrt{20}\left[(1/2)t^2\right]_1^2=\sqrt{20}(1/2)=4\sqrt{5}/2=2\sqrt{5}$ units

2. Compute the length of the arc of the curve $x=e^t \cos 3t$, $y=e^t \sin 3t$ from $t=0$ to $t=3$.

Solution:

$x=e^t\cos 3t$, $y=e^t\sin 3t$

$(dx/dt)= \longrightarrow$, $(dy/dt) = \longrightarrow$

$(dx/dt)=e^t\cos 3t+e^t(-3\sin 3t)=e^t(\cos 3t-3\sin 3t)$

$(dy/dt)=e^t\sin 3t+e^t(3\cos 3t)=e^t(\sin 3t+3\cos 3t)$

$(dx/dt)^2+(dy/dt)^2 = \longrightarrow$

$(dx/dt)^2+(dy/dt)^2=e^{2t}(\cos 3t-3\sin 3t)^2+e^{2t}(\sin 3t+3\cos 3t)^2$

$=e^{2t}(\cos^2 3t-6\sin 3t\cos 3t+9\sin^2 3t+\sin^2 3t$

$6\sin 3t\cos 3t+9\cos^2 3t)$

$=e^{2t}(1+9)$, (since $\sin^2 x+\cos^2 x=1$), $=10e^{2t}$

The arc length is

$\int_0^3 \sqrt{10e^{2t}}\,dt = \longrightarrow$

$$\int_0^3 \sqrt{10e^{2t}}\, dt = \sqrt{10} \int_0^3 e^t\, dt + 10 \left[e^t\right]_0^3 = \sqrt{10}e^3$$

The arc length is $10e^3$ units.

3. Find the length of the arc of the curve $y^5 = 3x^2$ from $x=1$ to $x=2$.

Solution:

$y^2 = 3x^2$, $dy/dx = \longrightarrow$

$2y\ dy/dx = 6x$

$dy/dx = 3x/y$

From $y^2 = 3x^2$, $y = \sqrt{3}x$

Therefore, $dy/dx = 3x/\sqrt{3}x = 3/\sqrt{3}$

$1+(dy/dx)^2 = \longrightarrow$

$1+(dy/dx)^2 = 1+9/3 = 1+3=4$

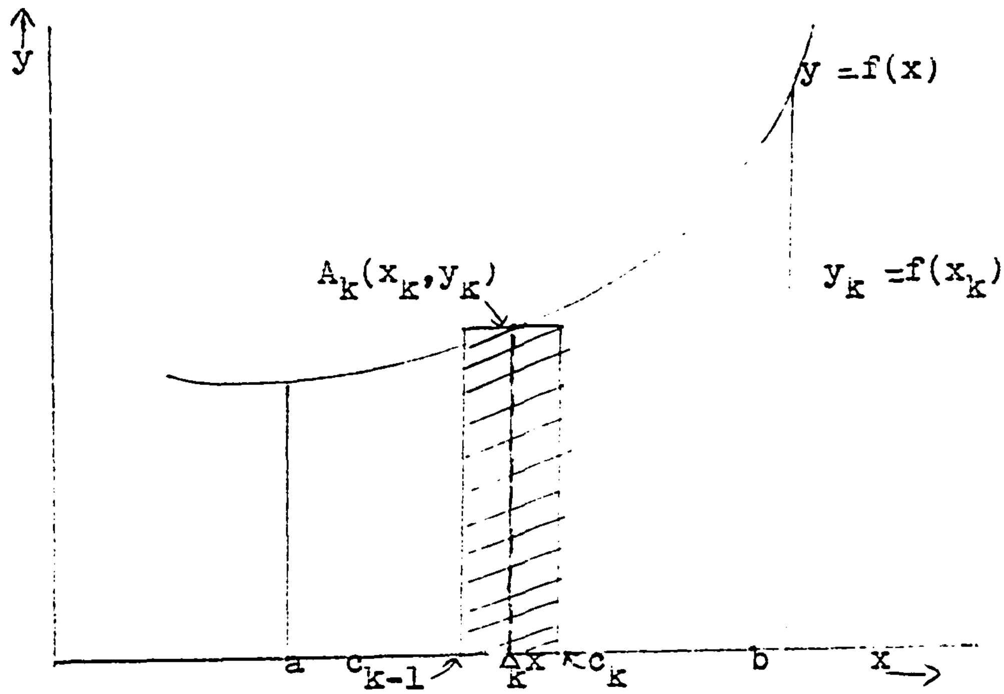
The arc length $s = \int_1^2 4dx = \longrightarrow$

The arc length $s = \int_1^2 4dx = \left[4x\right]_1^2$

$= 8-4 = 4$ units.

L. Unit 12

Computation of Cross-Sectional Area



Let f(x) be continuous and non-negative on the interval $a \leq x \leq b$ then

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(x_k) \Delta_k x = \int_a^b f(x)\, dx.$$

Divide the interval $a \leq x \leq b$ by points $c_0 = a, c_1, c_2 \ldots\ldots, c_n = b$. As shown in the figure, the area of the representative strip is $f(x_k) \Delta_k x$.

$$f(x_1) \Delta_1 x + f(x_2) \Delta_2 x + f(x_3) \Delta_3 x + \ldots + f(x_n) \Delta_n x$$

$= \sum_{k=1}^{n} f(x_k) \Delta_k x$ is the sum of the approximating rectangles.

The limit of this sum is

$$\lim_{n\to\infty} \sum_{k\,1}^{n} f(x_k)\,\Delta_k x = \int_a^b f(x)\,dx$$

which is also the area under the curve from $x=a$ to $x=b$.

To compute the area bounded by the curve, it is advisable to sketch the representative strip, and the area sought. Then write the area of the approximating rectangle and the sum for the n rectangles. Then apply the idea that

$$\lim_{n\to\infty} \sum_{k\,1}^{n} f(x_k)\,\Delta_k x = \int_a^b f(x)\,dx.$$

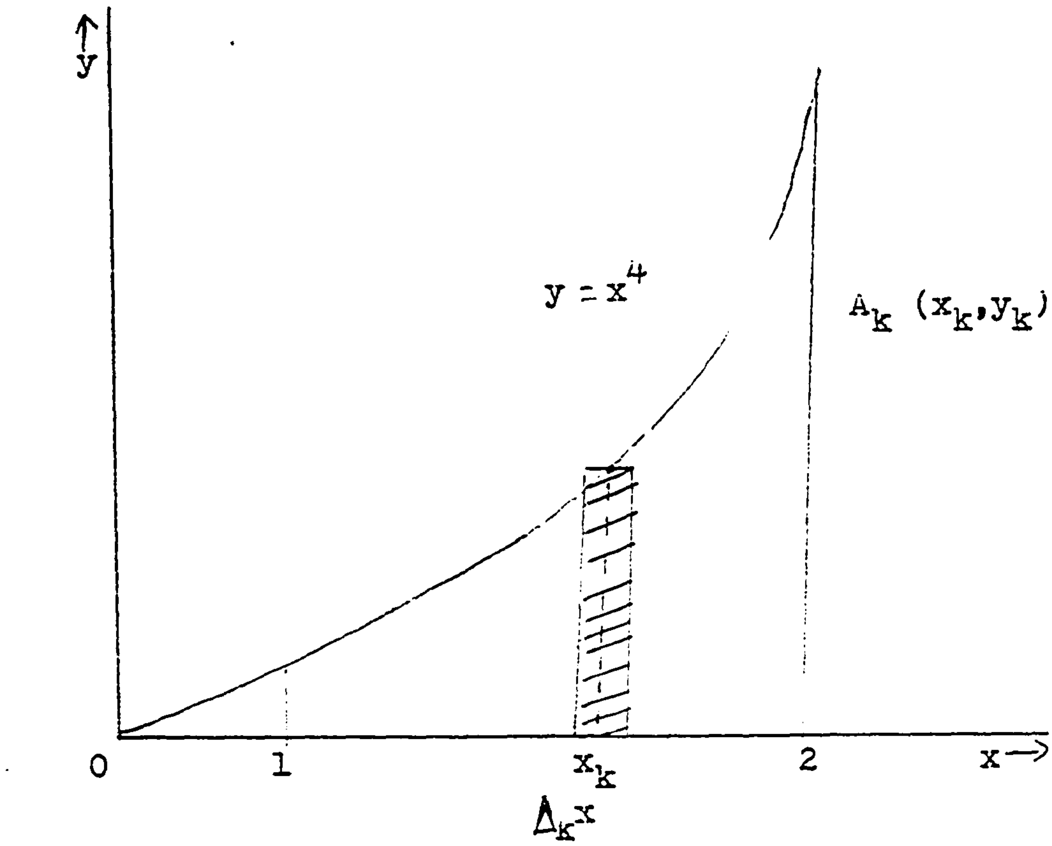
As one's skill is increased in solving the problems, some of the steps can be skipped.

Solved problem

Find the area bounded by the curve $y=x^2$, the x-axis and the ordinates $x=1$ and $x=2$.

Solution:

From the elementary strip in the figure on the next page, the base of the rectangle is $\Delta_k x$, the altitude is $y_k = f(x_k) = x_k^4$ and the area is $x_k^4 \, \Delta_k x$



Then the area bounded by the curve is

$$A = \lim_{n} \sum_{k\,1}^{n} x_4^k \,\Delta_k x = \int_1^2 x^4 \, dx$$

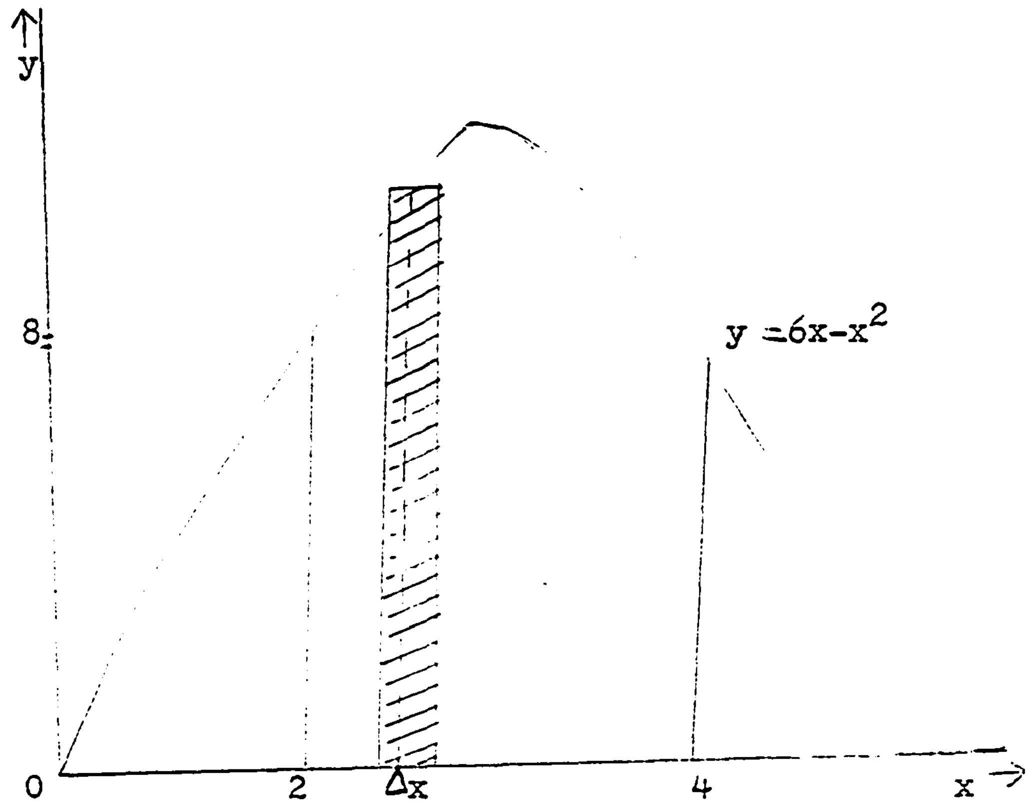
$$\left[x^5/5 \right]_1^2 = (1/5)(32-1) = (31/5)$$

The area bounded by the curve is 31/5 square units.

Now try solving the following problems.

1. Find the area bounded by the parabola $y = 6x - x^2$, the x-axis and the ordinates $x = 2$ and $x = 4$.

Solution:



From the elementary strip in the figure above, the

base of the rectangle is $\longrightarrow$

The base of the rectangle is Δx

The altitude of the rectangle is $\longrightarrow$

The altitude of the rectangle is $6x-x^2$

The area of the rectangle is $\longrightarrow$

The area of the rectangle is $(6x-x^2)\,\Delta x$

The required area is $\longrightarrow$

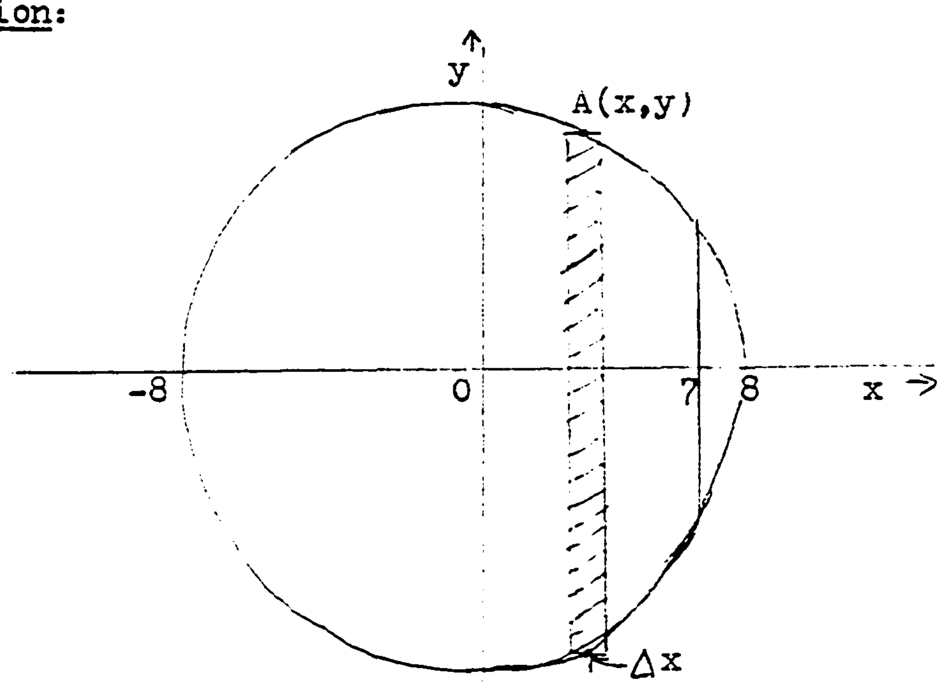
The required area is $\int_2^4 (6x-x^2)\,dx$

$= \left[3x^2-x^3/3\right]_2^4 = (48-64/3-12+8/3) = 52/3$

$= 52/3$ square units.

2. Find the larger area cut from the circle $x^2+y^2=64$ by the line $x=7$.

<u>Solution:</u>



From the elementary strip in the figure above, the base of the rectangle is $\longrightarrow$

The base of the rectangle is Δx

The altitude of the rectangle is $\longrightarrow$

The altitude of the rectangle is $2\sqrt{64-x^2}$

The area of the rectangle is $\longrightarrow$

The area of the rectangle is $2\sqrt{64-x^2}\,\Delta x$

The required area is $\longrightarrow$

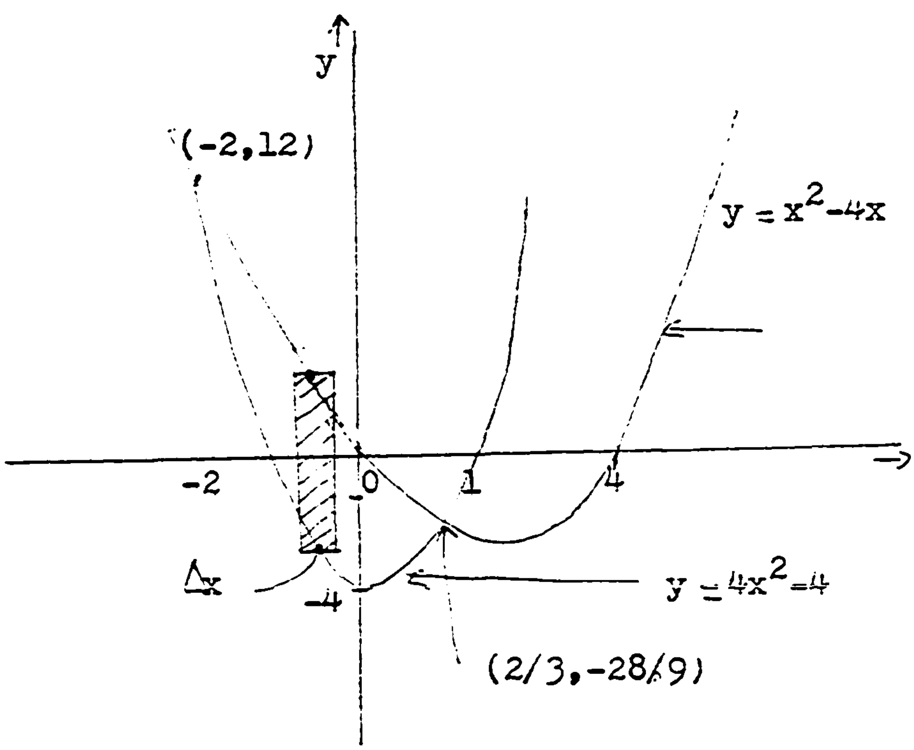
The required area is $\int_{-8}^{7} 2\sqrt{64-x^2}\,dx$

$$=2\left[(x/2)\sqrt{64-x^2}+(64/2)\text{arc }\sin(x/8)\right]_{-8}^{7}$$

$$=2((7/2)\sqrt{64-49}+32\text{arcsin}(7/8)+4\sqrt{64-64}+32\text{arc }\sin(-8/8))$$

$$=\left[7\sqrt{15}+64\text{arc }\sin(7/8)+64\text{arc }\sin(-1)\right]\text{units.}$$

3. Find the area bounded by the parabolas $y\ \ 4x^2-4$ and $y\ \ x^2-4x$



From the elementary figure in the figure above, the base of the rectangle is $\longrightarrow$

The base of the rectangle is Δx.

The altitude of the rectangle is $\longrightarrow$

The altitude of the rectangle is $x^2-4x-(4x^2-4)$

$$= -3x^2-4x+4$$

The area of the rectangle is $\longrightarrow$

The area of the rectangle is $(-3x^2-4x+4)\,\Delta x$

To obtain the points of intersection of the two parabolas, set $x^2-4x = 4x^2-4$

Hence, $3x^2+4x-4 = 0$

$$x = (-4 \pm \sqrt{16-4(3)(-4)})/6$$

$$x = (-4 \pm \sqrt{16+48})/6 = (-4 \pm \sqrt{64})/6$$

$$= (-4 \pm 8)/6$$

$$x = (-4+8)/6 = 4/6 = 2/3$$

$$x = (-4-8)/6 = -12/6 = -2$$

The required area is $\longrightarrow$

The required area is $\int_{-2}^{2/3} (-3x^2-4x+4)\,dx$

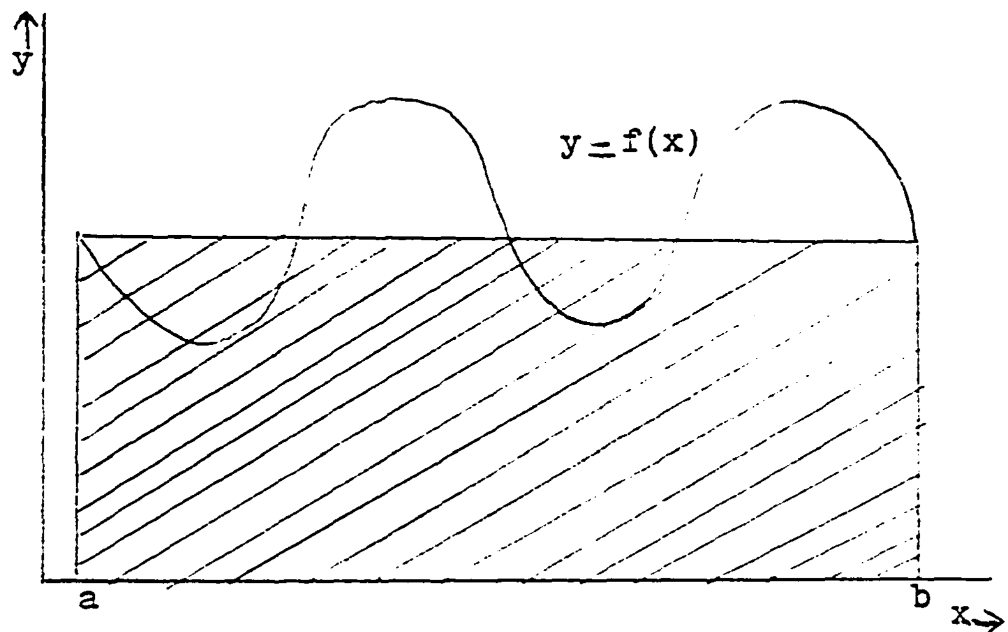
$$= \Big[-x^3-2x^2+4x\Big]_{-2}^{2/3} = -8/27-8/9+8/3-8+8+8$$

$$= 229/27 \text{ square units.}$$

M. Unit 13

The Average of a Function over an Interval

The average of a function $f(x)$ over an interval $[a,b]$ is given by
$\left(\int_a^b f(x)\,dx\right)/(b-a)$



The height of the rectangle is the the average value of $f(x)$ over $[a,b]$. The area of the rectangle is equal to the area of the region under the graph of $f(x)$.

Solved problem:

Compute the average value of $3\cos x$ in the interval $[0, \pi/2]$.

Solution:

The average value of the function is

$(\int_0^2 3\cos x \, dx)/(\pi - 0)$

$$\int_0^{\pi} 3\cos x \, dx = \Big[3\sin x\Big]_0^2 = 3(1) = 3$$

The average value of the function is

$3/(\pi - 0) = 3/\pi$.

Now try solving the following problems.

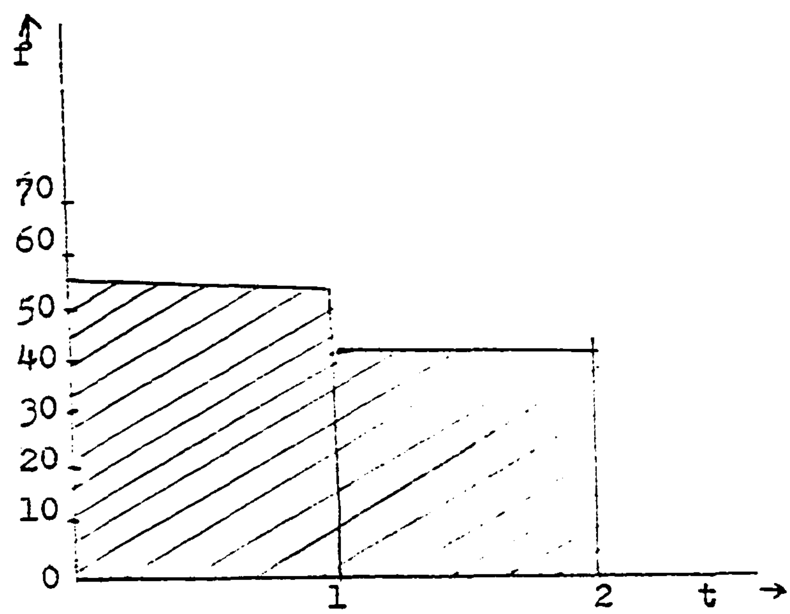
1. If a man travels at 55 miles per hour for 1 hour and 35 miles per hour for another hour, what is his average velocity with respect to time?

<u>Solution</u>:

Let f(t) be the velocity at time t.

The average velocity is $(\int_0^2 f(t)\, dt)/(2-0)$

Examine the following diagram



The area of the shaded region is $\longrightarrow$

The area of the shaded region is $\int_0^2 f(t)\,dt$

$= 55 + 35 = 90$

The average velocity is $\left(\int_0^2 f(t)\,dt\right)/(2-0) = 90/2$

$= 45$ miles per hour.

2. Compute the average value of $30x(1-\sqrt{x})^2$

in the interval $[0,1]$.

Solution:

The average value required is

$$\left(\int_0^1 30x(1-\sqrt{x})^2\,dx\right)/(1-0)$$

$$30\int_0^1 x(1-\sqrt{x})^2 = \longrightarrow$$

$$30\int_0^1 x(1-\sqrt{x})^2 = 30\int_0^1 (x-2x^{3/2}+x^2)\,dx$$

$$30\left[x^2/2-(4/5)x^{5/2}+x^3/3\right]_0^1 = 30\left[1/2-4/5+1/3\right]$$

$$= 30(1/30) = 1$$

The average value required is $\longrightarrow$

The average value required is $1/1 = 1$.

3. Compute the average value of e^{-3x} in the interval $[1,2]$.

Solution:

The average value required is

$$\left(\int_1^2 e^{-3x}\,dx\right)/(2-1)$$

$$\int_1^2 e^{-3x}\,dx = \longrightarrow$$

$$\int_1^2 e^{-3x}\,dx = \left[-e^{-3x}/3\right]_1^2 = (1/3\left[-e^{-6}+e^{-3}\right]$$

The average value required is $\longrightarrow$

The average value required is $(1/3)\left[e^{-6}+e^{-3}\right]$

$= (1/3)(e^{-3}-e^{-6})$.

N. Unit 14

Improper Integrals

$\int_a^b f(x)\,dx$ is an improper integral if $f(x)$ has at least one point of discontinuity in the interval $a \leq x \leq b$ or at least a or b is infinite.

Case 1: $f(x)$ is discontinuous at some point.

Let $f(x)$ be continuous on $a \leq x \leq b$ and discontinuous at $x = b$, then

$$\int_a^b f(x)\,dx = \lim_{\epsilon \to 0^+} \int_a^{b-\epsilon} f(x)\,dx \text{ if the limit exists.}$$

Let $f(x)$ be continuous on $a < x \leq b$ and discontinuous at $x = a$, then

$$\int_a^b f(x)\,dx = \lim_{\epsilon \to 0^+} \int_a^b f(x)\,dx \text{ if the limit exists.}$$

Let $f(x)$ be continuous on $a \leq x \leq b$ and discontinuous at $x = c$ where $a < c < b$, then

$$\int_a^b f(x)\,dx = \lim_{\epsilon \to 0^+} \int_a^{c-\epsilon} f(x)\,dx + \lim_{\delta \to 0^+} \int_{c+\delta}^b f(x)\,dx,$$

if the limit exists.

Case 2: At least one of the limit points is infinite

Let $f(x)$ be continuous on $a \leq x \leq s$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{s\to\infty} \int_a^s f(x)\,dx \text{ if the limit exists.}$$

Let $f(x)$ be continuous on $t \leq x \leq b$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to\infty} \int_t^b f(x)\,dx \text{ if the limit exists.}$$

Let $f(x)$ be continuous on $t \leq x \leq s$, then

$$\int_{-\infty}^{+\infty} f(x)\,dx \quad \lim_{t\to\infty} \int_c^s f(x)\,dx + \lim_{s\to -\infty} \int_s^c f(x)\,dx$$

if both limits exist.

<u>Solved problem</u>:

Evaluate $\int_2^{\infty} (1/x^3)\,dx$

<u>Solution</u>:

This problem involves case 2 in which the upper limit is infinite.

$$\int_2^{\infty} (1/x^3)\,dx = \lim_{t\to\infty} \int_2^t x^{-3}\,dx = \lim_{t\to\infty} \Big[-x^{-2}/2\Big]_2^t$$

$$= \lim_{t\to\infty} (-1/2t^2 + 1/8) = 1/8$$

Therefore,

$$\int_2^{\infty} (1/x^3)\,dx = 1/8.$$

Now try solving the following problems.

1. Find $\int_0^{\infty} e^{-4x} \cos x\,dx$

Solution:

This problem involves case 2 in which the upper limit is infinite.

$$\int_0^{\infty} e^{-4x}\cos x\ dx = \lim_{t\to\infty} \int_0^t e^{-4x}\ \cos x\ dx$$

Using the method of integration by parts,

$$\int e^{-4x}\ \cos x\ dx = \longrightarrow$$

$$\int e^{-4x}\ \cos x\ dx = e^{-4x}\sin x + 4\int e^{-4x}\sin x\ dx$$

$$= e^{-4x}\sin x\ 4\left[-e^{-4x}\cos x\ -\ 4\int e^{-4x}\cos x\ dx\right]$$

$$\int e^{-4x}\cos x\ dx = e^{-4x}\sin x - 4e^{-4x}\cos x - 16\int e^{-4x}\cos x\ dx$$

$$17\int e^{-4x}\cos x\ dx = e^{-4x}(\sin x - 4\cos x)$$

$$\int e^{-4x}\cos x\ dx = e^{-4x}(\sin x - 4\cos x)/17$$

$$\int_0^t e^{-4x}\cos x\ dx = \longrightarrow$$

$$\int_0^t e^{-4x}\cos x\ dx = \left[e^{-4x}\ (\sin x - 4\cos x)/17\right]_0^t$$

$$= e^{-4t}(\sin t - 4\cos t)/17 + 4/17$$

$$\lim_{t\to\infty}\int_0^t e^{-4x}\cos x\ dx = \longrightarrow$$

$$\lim_{t\to\infty}\int e^{-4x}\cos x\ dx = \lim_{t\to\infty}\ e^{-4t}(\sin t - 4\cos t) + 4/17 = 4/17$$

Therefore, $\int_0^{\infty} e^{-4x}\cos x\, dx = \longrightarrow$

$$\int_0^{\infty} e^{-4x}\cos x\, dx = \lim_{t\to\infty}\int_0^{t} e^{-4x}\cos x\, dx = 4/17.$$

2. Evaluate $\int_{-\infty}^{2} e^{x}\, dx.$

Solution:

This problem involves case 2 in which the lower limit is infinite.

$$\int_{-\infty}^{2} e^{x}\, dx = \lim_{t\to-\infty}\int_{t}^{2} e^{x}\, dx.$$

$$\int_{t}^{2} e^{x}\, dx = \left[e^{x}\right]_{t}^{2} = e^{2}-e^{t}$$

$$\lim_{t\to-\infty}\int_{t}^{2} e^{x}\, dx = \longrightarrow$$

$$\lim_{t\to-\infty}\int_{t}^{2} e^{x}\, dx = \lim_{t\to-\infty}\left[e^{2}-e^{t}\right] = e^{2}$$

Therefore, $\int_{-\infty}^{2} e^{x}\, dx = \longrightarrow$

$$\int_{-\infty}^{2} e^{x}\, dx = \lim_{t\to-\infty}\left[e^{2}-e^{t}\right] = e^{2}.$$

3. Find $\int_0^{\infty} 2/(1+x^2)\, dx$

<u>Solution</u>:

This problem involves case 2 in which the upper limit is infinite.

$$\int_0^{\infty} 2/(1+x^2)\ dx \quad \lim_{t\to\infty} \int_0^{t} 2/(1+x^2)\ dx$$

$$\int_0^{t} 2/(1+x^2)\ dx = \longrightarrow$$

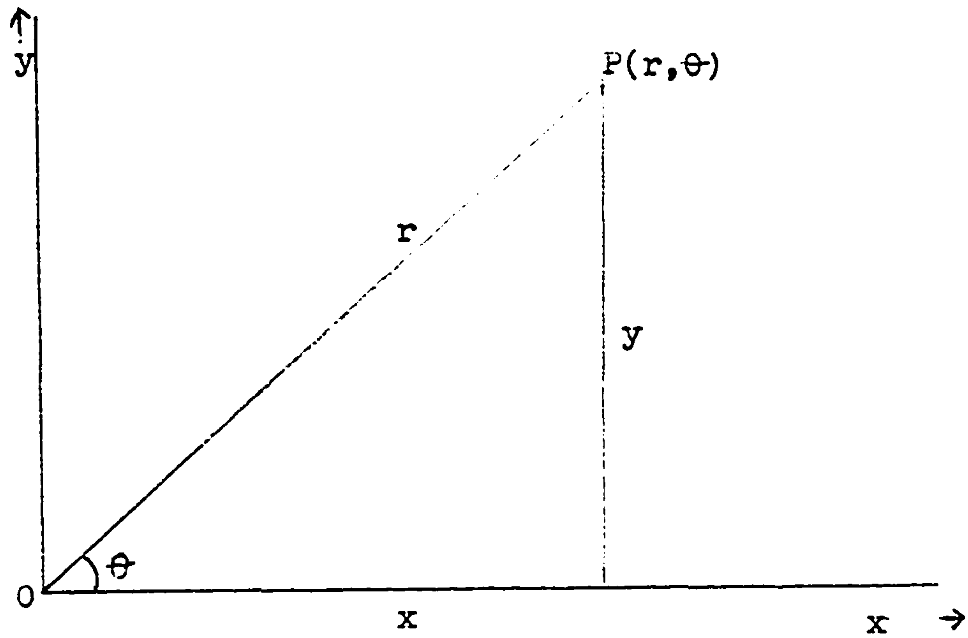
$$\int_0^{t} 2/(1+x^2)\ dx = \Big[2\text{arc tan}x\Big]_0^t = 2\text{arc tan}t$$

$$\lim_{t\to\infty} \int 2/(1+x^2)\ dx = \longrightarrow$$

$$\lim_{t\to\infty} \int 2/(1+x^2)\ dx = \lim_{t\to\infty} 2\ \text{arc tan}t = 2(\pi/2) = \pi.$$

O. Unit 15

Polar Coordinates



Let $OP = r$ make an angle θ with OX, then the polar coordinates of the point P is (r, θ). The rectangular coordinates of the point P are $(r,\theta) = (x,y)$. The point O is called the pole while OX is called the polar axis. The relations between rectangular and polar coordinates are

$x = r\cos\theta$, $y = r\sin\theta$

and $r^2 = x^2 + y^2$, $\tan\theta = y/x$.

Solved problem:

Express $(2, 5\pi/6)$ in rectangular coordinates

Solution:

Using the notation in this unit, $r = 2$ and

$\theta = 5\pi/6$.

The objective is to find out the value of x and y. From the relations between rectangular and polar coordinates,

$x = r\cos\theta = 2\cos(5\pi/6) = 2(-\sqrt{3}/2) = -\sqrt{3}$

$y = r\sin\theta = 2\sin(5\pi/6) = 2(1/2) = 1.$

Therefore, the rectangular coordinates of the point are $(-\sqrt{3},1)$

Now try solving the following problems

1. Express $(-4, 3\pi/4)$ in rectangular coordinates.

<u>Solution</u>:

Using the notation in this unit,

$r = \longrightarrow$ and $\theta = \longrightarrow$

$r = -4$, $\theta = 3\pi/4$

The objective is to find out the value of x and y. From the relations between rectangular and polar coordinates,

$x = \longrightarrow$ and $y = \longrightarrow$

$x = r\cos\theta = -4\cos(3\pi/4) = -4(-\sqrt{2}/2) = 2\sqrt{2}$

$y = r\sin\theta = -4\sin(3\pi/4) = -4(\sqrt{2}/2) = -2\sqrt{2}$

Therefore the rectangular coordinates of the point are $\longrightarrow$

The rectangular coordinates of the point are

$(2\sqrt{2},-2\sqrt{2})$.

2. Express $(-1,-1)$ in polar coordinates.

<u>Solution</u>:

Using the notation in this unit,

$x = \longrightarrow$ and $y = \longrightarrow$

$x = -1$, $y = -1$

The objective is to find out the value of r and θ.

From the relations between rectangular and polar coordinates,

$r = \longrightarrow$ and $\theta = \longrightarrow$

$r^2 = x^2 + y^2 = (-1)^2 + (-1)^2 = 1+1 = 2$

$r = \sqrt{2}$

$\tan\theta = y/x = -1/-1 = 1$

The required angle θ is in the third quadrant.

$\theta = \arc\tan 1 = 5\pi/4$

Therefore the polar coordinates of the point are $\longrightarrow$

The polar coordinates of the point are $(\sqrt{2}, 5\pi/4)$.

3. Transform the equation $y = 3x+1$ into polar coordinates.

<u>Solution</u>:

Using the notation in this unit,

$x = \longrightarrow$ and $y = \longrightarrow$

$x = r\cos\theta$ and $y = r\sin\theta$

The equation becomes

$r\sin\theta = 3r\cos\theta + 1$

$r(\sin\theta - 3\cos\theta) = 1$

$r = 1/(\sin\theta - 3\cos\theta)$

P. Unit 16

Infinite Sequences and Series

Definitions on sequences

If the domain of a function consists of positive integers $\{a_n\} = a_1, a_2, a_3, \ldots, a_n \ldots$ then it is called an infinite sequence.

A sequence $\{a_n\}$ is bounded if there exists numbers A and B such that $A \leq a_n \leq B$ for $n = 1, 2, \ldots$

A sequence $\{a_n\}$ is nonincreasing if $a_1 \geq a_2 \geq a_3 \ldots \geq a_n$. A sequence $a_n\}$ is nondecreasing if $a_1 \leq a_2 \leq a_3 \ldots \leq a_n$.

A sequence $\{a_n\}$ converges to a, that is $\lim_{n\to\infty} a_n = a$, if for any positive small number ϵ, there exists a positive number N such that whenever $n > N$, then $|a - a_n| < \epsilon$. A sequence with a limit is a convergent sequence while a sequence without a limit is called a divergent sequence.

A sequence $\{a_n\}$ diverges to ∞, that is $\lim_{n\to\infty} a_n = \infty$, if for any large positive number M, there exists a positive integer m such that whenever $n > m$ then $|a_n| > M$. If $a_n > M$, $\lim_{n\to\infty} a_n = \infty$ but if $a_n < -M$, $\lim_{n\to\infty} a_n = -\infty$.

<u>Definations on sums</u>

An infinite sequence $\{a_n\}$ is called an infinite series if

$$\sum a_n = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots a_n + \ldots$$

An associated sequence of partial sums of the series is $S_1 = a_1$, $S_2 = a_1 + a_2$, $S_3 = a_1 + a_2 + a_3 \ldots\ldots\ldots\ldots$,

$S_n = a_1 + a_2 + a_3 + \ldots + a_n$.

The series $\sum a_n$ converges to S, its partial sum if $\lim_{n\to\infty} S_n = S$.

The series $\sum a_n$ diverges if $\lim_{n\to\infty} S_n$ does not exist.

Important theorems in sums and sequences

Assume $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$

1. $\lim_{n\to\infty} (ca_n) = c \lim_{n\to\infty} a_n = ca$, where c is a constant

2. $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = a \pm b$

3. $\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n = ab$

   $\lim_{n\to\infty} (a_n / b_n) = \lim_{n\to\infty} a_n / \lim_{n\to\infty} b_n = a/b$

4. If $\sum a_n$ converges to A, then $\sum c\, a_n$ converges to c A where c is a constant .

5. If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. The converse is false since for the series $\sum 1/n$, $\lim_{n\to\infty} 1/n = 0$ but $\sum 1/n$ diverges.

6. If $\lim_{n\to\infty} a_n \neq 0$, the $\sum a_n$ diverges.

7. If $-1 < r < 1$, the geometric series $a + ar + \ldots + ar^{n-1} + \ldots$ converges to $a/(1-r)$. a is the first term, and r is the common ratio.

<u>Proof</u>: Let S_n the sum of the first n terms.

$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$.

$rS_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n$.

Therefore, $S_n - rS_n = a - ar^n$

$(1-r)\ S_n = a(1-r^n)$

$S_n = a(1-r^n)/(1-r)$

$$\lim_{n\to\infty} S_n = a/(1-r) - 1/(1-r) \lim_{n\to\infty} (r^n) = a/(1-r)$$

$$\lim_{n\to\infty} S_n = a/(1-r)$$

<u>Solved problem</u>

Find out if the sequence $\{2-2/n\}$ converges or not. If the sequence converges then compute what it converges to.

<u>Solution</u>:

The n^{th} term is $a_n = 2-2/n$

$a_{n+1} = 2-2/(n+1) = 2-2/n+2/n(n+1) = a_n + 2/n(n+1)$

$\therefore \quad a_{n+1} \geq a_n$ and the sequence is nondecreasing.

Also for all n, $0 \leq a_n \leq 2$. a_n is bounded.

Since the sequence is bounded and non-decreasing, it is convergent. The sequence converges to 2.

Now try solving the following problems:

1. Is the series $\sum_{n+1}^{\infty} 1/\ n$ convergent or divergent?

If the series is convergent, what number does it converge to?

<u>Solution</u>:

The sum of n terms is $S_n = 1/\sqrt{1} + 1/\sqrt{2} + \dots\dots + 1/\sqrt{n}$

But $1/\sqrt{1} \geq 1/\sqrt{n}$, $1/\sqrt{2} \geq 1/\sqrt{n}$, $1/\sqrt{3} \geq 1/\sqrt{n}$, ...

Hence $S_n \geq 1/\sqrt{n} + 1/\sqrt{n} + \dots + 1/\sqrt{n} = n/\sqrt{n} = \sqrt{n}$

That is $S_n \geq \sqrt{n}$

$\lim_{n \to \infty} S_n \geq \lim_{n \to \infty} \sqrt{n} = \longrightarrow$

$\lim_{n \to \infty} S_n = \infty$

$\sum_{n=1}^{\infty} (1/\sqrt{n})$ is $\longrightarrow$

$\sum_{n\ 1}^{\infty} (1/\sqrt{n})$ is divergent.

2. Find $\sum_{n=1}^{\infty} (0.12)^n$

Solution:

$$\sum_{n=1}^{\infty} (0.12)^n = 0.12 + 0.12^2 + \ldots + (0.12)^n + \ldots$$

The first term a is $\longrightarrow$

The first term a is 0.12

The common ratio r is $\longrightarrow$

The common ratio r is 0.12

The sum of the series is $a/(1-r) = \longrightarrow$

The sum of the series is $0.12/(1-0.12) = 0.12/0.88$

$= 0.03/0.02 = 3/22$.

$$\sum_{n=1}^{\infty} 0.12^n = 3/22.$$

3. Prove that if $c > 1$, then $\lim_{n\to\infty} c^n = \infty$.

Solution:

Choose $M > 0$. Let $1 = 1+k$, where $k > 0$.

Expanding by the binomial theorem,

$c^n = (1+k)^n = \longrightarrow$

$c^n = (1+k)^n = 1+nk+n(n-1)k^2/2\ldots > 1+nk > M$

where $n > M/k$. A suitable m is the largest in M/k.

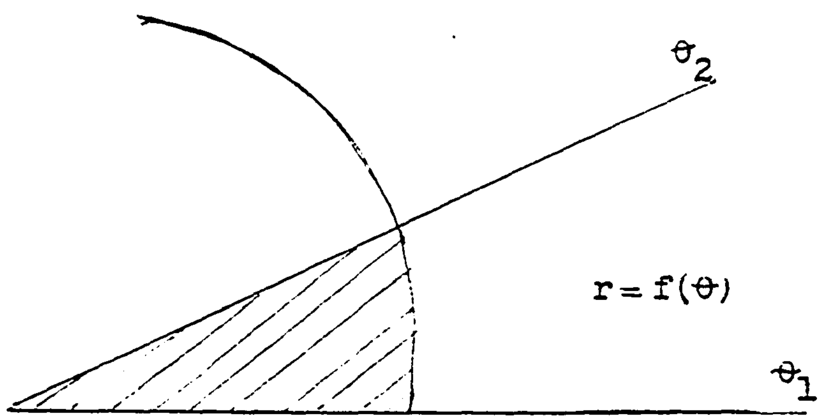
$\lim_{n} c^n > \lim_{n\to\infty} 1+nk = \longrightarrow$

Therefore, $\lim_{n\to\infty} c^n = \infty$.

Q. Unit 17

Area in Polar Coordinates

Let the area be bounded by the radius vectors $\theta = \theta_1$, and $\theta = \theta_2$.



The plane area bounded by the curve $r = f(\theta)$ and the radius vectors $\theta = \theta_1$ and $\theta = \theta_2$ is given by

$$(1/2) \int_{\theta_1}^{\theta_2} r^2 \, d\theta.$$

Solved problem:

Find the area bounded by $r = \cos 2t$ bounded by $t = 0$ and $t = \pi/4$.

Solution:

The required area is $\int_0^{\pi/4} (\cos 2t)^2/2 \, dt$

$$\int_0^{\pi/4} \cos^2 2t/2 \, dt = \int_0^{\pi/4} (\cos 4t + 1)/4 \, dt$$

$$= \left[(\sin 4t)/4 + t \right]_0^{\pi/4} /4 = \left[(\sin \pi)/4 + \pi/4 - \sin 0 - 0 \right] /4$$

$= \pi/16$

Therefore, $\int_0^{\pi/4} (\cos 2t)^2/2 \, dt = \pi/16$

Now try solving the following problems

1. Find the area of the region bounded by $r = 8\theta$ $\theta = \pi/8$ to $\theta = \pi/4$.

Solution:

The area required is $1/2 \int_{\pi/8}^{\pi/4} r^2 \, d\theta = 1/2 \int_{\pi/8}^{\pi/4} (8\theta)^2 \, d\theta$

$$= (1/2) \int_{\pi/8}^{\pi/4} 64\theta^2/2 \, d\theta = (1/2) \left[(64/3)\theta^3 \right]_{\pi/8}^{\pi/4}$$

$$= (64/6) \left[\pi^3/64 - \pi^3/192 \right] = 32\pi^3/3 \left[(3-1)/192 \right]$$

$$= (32\pi^3/2)(2/192)$$

$$= \pi^3/9.$$

Hence, the area required is $\pi^3/9$.

2. Compute the area of the region bounded by, $r = \sec\theta$, $\theta = 0$ and $\theta = \pi/4$.

Solution:

The area required is $(1/2) \int_0^{\pi/4} \sec^2\theta \, d\theta$

$$= (1/2) \left[\tan\theta \right]_0^{\pi/4} = (1/2) \left[\tan(\pi/4) - \tan \theta \right]$$

$$= (1/2)(1) = 1/2 .$$

Hence the required area is 1/2.

3. Find the area of the region bounded by

$r = 3 + \cos\theta$, $\theta = 0$, and $\theta = \pi$.

<u>Solution</u>:

The required area is $\int_0^{\pi} (3+\cos\theta)^2/2 \, dO$

$$= \int_0^{\pi} (9+6\cos\theta+\cos^2\theta)/2 \, d\theta$$

$$= \int_0^{\pi} (1/2)(9+6\cos\theta+\cos 2\theta/2+1/2) \, d\theta$$

$$= \Big[9\theta + 6\sin\theta + (\sin 2\theta)/4 + \theta/2\Big]_0^{\pi}/2$$

$$= \Big[19\theta/2 + 6\sin\theta + (\sin 2\theta)/4\Big]/2$$

$$= (1/2)(19\pi/2) = 19\pi/4.$$

Hence the required area is $19\pi/4$.

R. Unit 18

Area of a Surface of Revolution

A short way to write the formula for the surface area of revolution is

$$\int_a^b 2\pi R ds \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where R is the radius of revolution, s the arc length and $[a,b]$ the interval.

Let $x = g(t)$, $y = h(t)$ be parametic equations of a curve. Also let g and h have continuous derivatives $h(t) \geq 0$. If Γ is the portion of the curve corresponding to t in $[a,b]$, then the area of the surface of revolution formed by revolving Γ about the x-axis is

$$\int_0^b 2\pi h(t) \sqrt{[g'(t)]^2 + [h'(t)]^2}\, dt \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Substituting $y = h(t)$, $dx/dt = g'(t)$ and $dy/dt = h'(t)$, the area of the surface of revolution is

$$\int_a^b 2\pi y \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Let a curve be given by y f(x), where f has a continuous derivative and $f(x) \geq 0$. Le the curve be parametized by the equations $x = t$, $y = f(t)$ then $dx/dt = 1$. The area of the surface area obtained by revolving the curve above $[a,b]$ about the x axis is

$$\int_a^b 2\pi y\sqrt{1+(dy/dx)^2}\,dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Solved problem

Find the area of the surface obtained by revolving part of the curve $y = x/2$ that lies between $x = 0$ and $x = 1$ about the x axis.

Solution:

Using the notation in this lesson,

$y = x/2$ and $dy/dx = 1/2$.

Apply formula 4 in this lesson. The surface area is given by

$$\int_a^b 2\pi y\sqrt{1+(dy/dx)^2}\,dx = \int_0^1 2\pi x/2\sqrt{1+1/4}\,dx$$

$$= \int_0^1 \pi x\sqrt{5/4}\,dx = \sqrt{5}\pi/2\int_0^1 x\,dx$$

$$\sqrt{5}\pi/2\left[x^2/2\right]_0^1 = \sqrt{5}\pi/2(1/2) = \sqrt{5}\pi/4$$

Hence the surface area is $\sqrt{5}\pi/4$.

Now try solving the following problems.

1. Find the area of the curve obtained by revolving part of the curve $y = x^3$ between $x = 0$ and $x = 2$ about the x-axis.

Solution:

Using the notation in this lesson,

$y = \longrightarrow$ and $dy/dx = \longrightarrow$

$y = x^3$ and $dy/dx = 3x^2$

Apply formula 4 in this lesson. The surface area is given by

$$\int_a^b 2\pi y\sqrt{1+(dy/dx)^2}\,dx = \longrightarrow$$

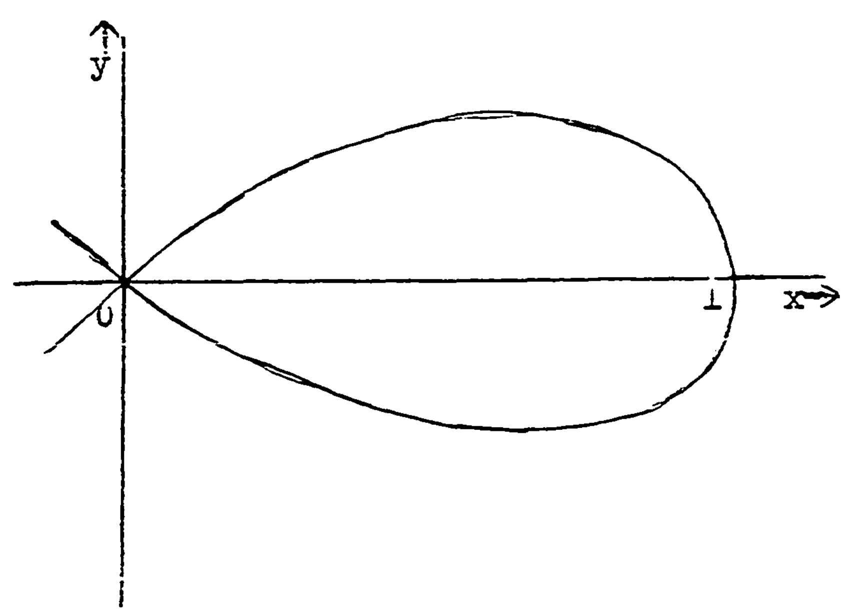
$$\int_a^b 2\pi y\sqrt{1+(dy/dx)^2}\,dx = \int_0^2 2\pi x^3\sqrt{1+(3x^2)^2}\,dx$$

$$= \int_0^2 2\pi x^3\sqrt{1+9x^4}\,dx = \pi\int_0^2 36x^3(\sqrt{1+9x^4})/18\,dx$$

$$= (2\pi/3)\Big[(1+9x^4)^{3/2}\Big]_0^2 = (2\pi/3)\Big[(145)^{3/2}-1\Big]$$

Hence the surface area is $2\pi\Big[(145)^{3/2}-1\Big]/3$.

2. Compute the area of the surface of revolution generated by revolving a loop of the curve $8y^2-x^2+x^4=0$ about the x-axis.



Solution:

Using the notation in the lesson

$y = \longrightarrow$ and $dy/dx = \longrightarrow$

$8y^2 - x^2 + x^4 = 0$

$y^2 = (x^2 - x^4)/8$

$y = (\sqrt{x^2 - x^4})/8 = (x\sqrt{1-x^2})/2\sqrt{2}$

$dy/dx = 1/(2\sqrt{2})\left[\sqrt{1-x^2} - x^2/\sqrt{1-x^2}\right]$

$= 1/2\sqrt{2}\left[(1-x^2-x^2)/\sqrt{1-x^2}\right] = (1/2\sqrt{2})\left[(1-2x^2)/\sqrt{1-x^2}\right]$

$1 + (dy/dx)^2 = 1 + (1-2x^2)^2/8(1-x^2)$

$= (8 - 8x^2 + 1 - 4x^2 + 4x^4)/8(1-x^2)$

$= (9 - 12x^2 + 4x^4)/8(1-x^2) = (3-2x^2)^2/(8(1-x^2))$

$2\pi\int_a^b y\sqrt{1+(dy/dx)^2}\, dx$

$= 2\pi\int_0^1 x\sqrt{1-x^2}(3-2x^2)^2/2\sqrt{2})(2\sqrt{2})\sqrt{1-x^2}$

$= (\pi/4)\int_0^1 (3-2x^2)x\, dx = \pi/4.$

The area of surface of revolution is $\pi/4$.

3. Find the area of the surface of revolution of

the curve $x = 2\cos^3 t$, $y = 2\sin^3 t$ about the x axis.

<u>Solution</u>:

Using the notation in this lesson,

$x = \longrightarrow$ and $dx/dt = \longrightarrow$

$x = 2\cos^3 t$ and $dx/dt = 6\cos^2 t \sin t$.

$y = \longrightarrow$ and $dy/dt = \longrightarrow$

$y = 2\sin^3 t$ and $dy/dt = 6\sin^2 t \cos t$.

$(dx/dt)^2 + (dy/dt)^2 = \longrightarrow$

$(dx/dt)^2 + (dy/dt)^2 =$

$= 36\cos^4 t \sin^2 t + 36\sin^4 t \cos^2 t$

$= 36\cos^2 t \sin^2 t(\cos^2 t + \sin^2 t)$

$= 36\cos^2 t \sin^2 t$

The required surface is generated by revolving from

$\theta = 0$ to $\theta = \pi$.

The area required is $2(2\pi)\int_0^{\pi/2} y\sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt$

$= \longrightarrow$

The area required is

$$4\pi\int_0^{\pi/2}(2\sin^3 t)(6\cos t\sin t)\,dt$$

$$= 48\pi\int_0^{\pi/2}\sin^4 t\cos t\,dt = (48\pi/5)\Big[\sin^5 t\Big]_0^{\pi/2}$$

$$= 48\pi(1/5) = 48\pi/5$$

Hence the surface required is $48\pi/5$.

S. Unit 19

Volume of a Solid of Revolution

If a plane area is revolved about a line called the axis of revolution, the volume of a solid of revolution is obtained. The two methods used to calculate the volume of a solid of revolution are the disc method and the shell method.

The disc method

If the axis of revolution is part of the area required, draw the area of the representative strip, write the volume obtained by rotating the representative strip and integrate to obtain the volume of the solid of revolution required. If the axis of revolution is not part of the area required, draw the area of the representative strip, extend the sides of the strip to meet the axis of rotation, write the volume obtained by rotating the representative strip and integrate to obtain the volume of the solid of revolution required.

The shell method

Draw the area of the representative strip, write the volume of the shell generated when the representative strip is revolved about the axis of revolution and integrate to obtain the volume of the solid of revolution required.

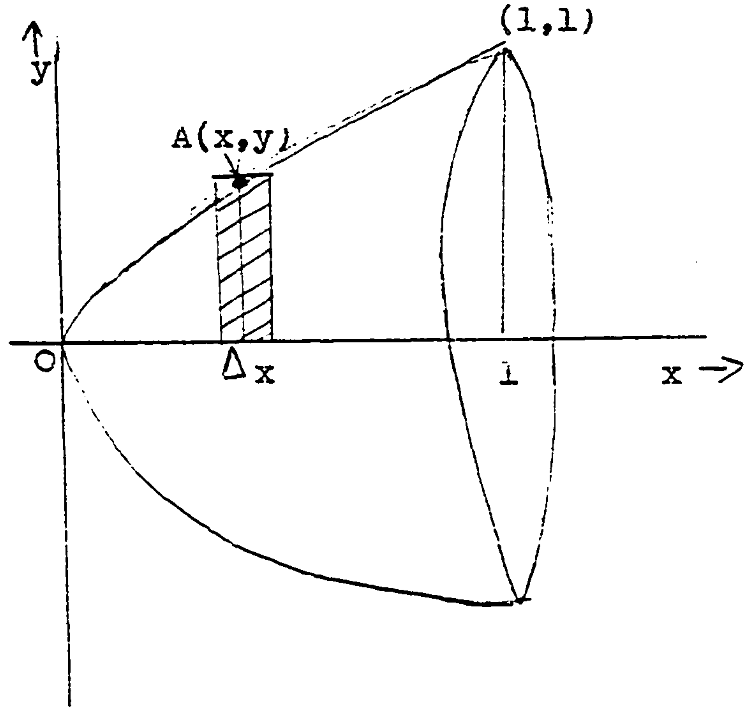
Solved problem:

Find the volume generated by revolving the first quadrant area bounded by the parabola

$y^2 = x$ and $x = 1$ about the x axis.

Solution:

The disc method could be used.



The volume obtained by generating the representative strip about the x axis is $\pi y^2 \Delta x$.

The volume of n approximating rectangles is $\sum y^2 \pi \Delta x$.

The required volume is

$$V = \int_0^1 \pi y^2 \, dx = \pi \int_0^1 x \, dx = \left[x^2/2 \right]_0^1 = \pi(1/2) = \pi/2.$$

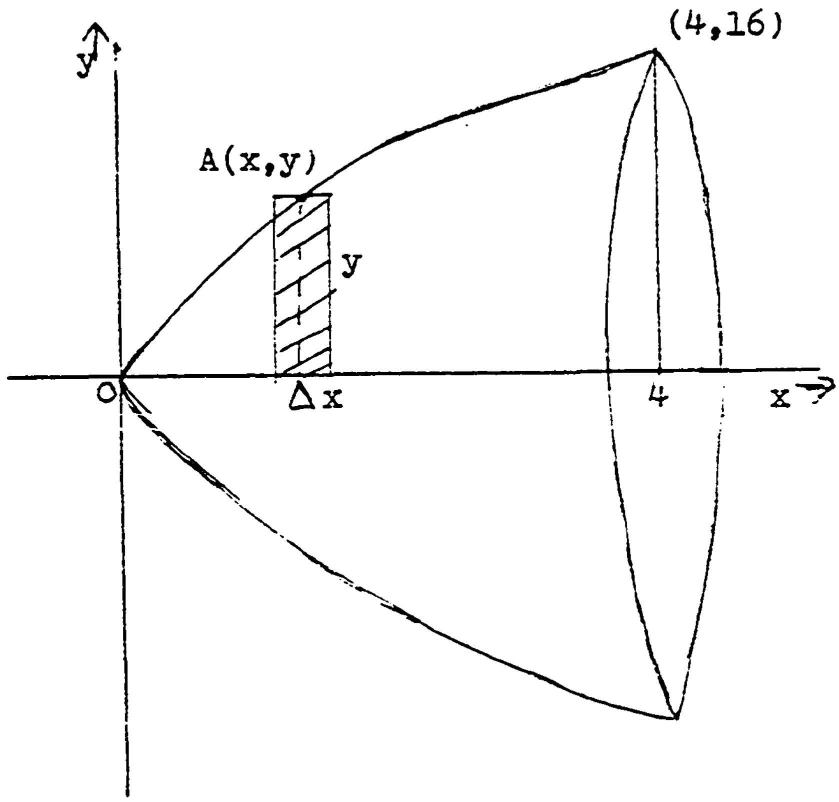
Hence the required volume is $\pi/2$ cubic units.

Now try solving the following problems.

1. Find the volume obtained by revolving the first quadrant area bounded by the parabola $y^2 = 4x$ and $x = 4$ about the x axis.

<u>Solution</u>:

Below is a sketch of the volume desired.



The volume obtained by generating the representative strip about the x axis is $\pi y^2 \Delta x$

The volume of n approximating rectangles is $\sum \pi y^2 \Delta x$.

The required volume is $\longrightarrow$

The required volume is $\int_0^4 y^2 \, dx = \pi \int_0^4 4x \, dx$

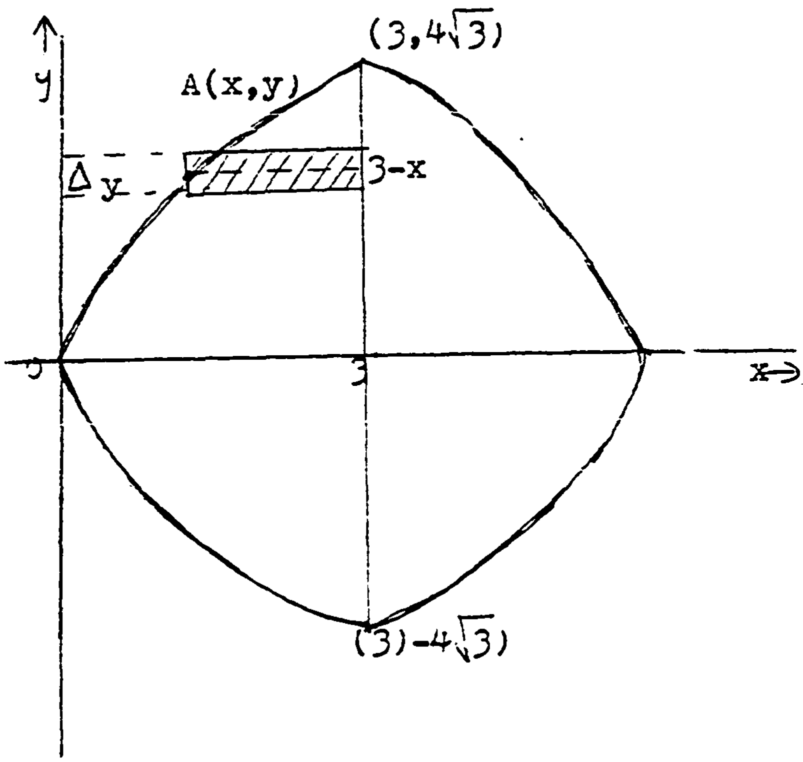
$= \pi \left[4x^2/2 \right]_0^4 = \pi 4(16)/2 = 32\pi$

Hence the required volume is 32π cubic units.

2. Use the disc method in calculating the volume generated by revolving the area bounded by $16x = y^2$ and $x = 3$ about the line $x = 3$.

<u>Solution</u>:

Below is a sketch of the volume desired.



When the representative strip is revolved about x =3 it generates a disc whose radius is $\longrightarrow$

The radius of the disc is 3- x.

The height of the disc is $\longrightarrow$

The height of the disc is Δy.

The volume of the disc is $\longrightarrow$

The volume of the disc is $\pi(3-x)^2$ y.

The required volume is $\longrightarrow$

The required volume is $\int_{-4\sqrt{3}}^{4\sqrt{3}} \pi (3-x)^2\, dy$

$$= \int_{-4\sqrt{3}}^{4\sqrt{3}} (3-y^2/16)^2\, dy = 2\pi \int_0^{4\sqrt{3}} (9-(3/8)y^2+y^4/256)\, dy$$

$$= 2\pi \left[9y-y^3/8+y^5/5(256)\right]_0^{4\sqrt{3}}$$

$$= 2\pi \left[36\sqrt{3}-(8)3^{3/2}+(4)3^{5/2}/5\right]$$

$$= 2\pi\sqrt{3}(36-24+36/5) = 2\pi\sqrt{3}\ (12+36/5)$$

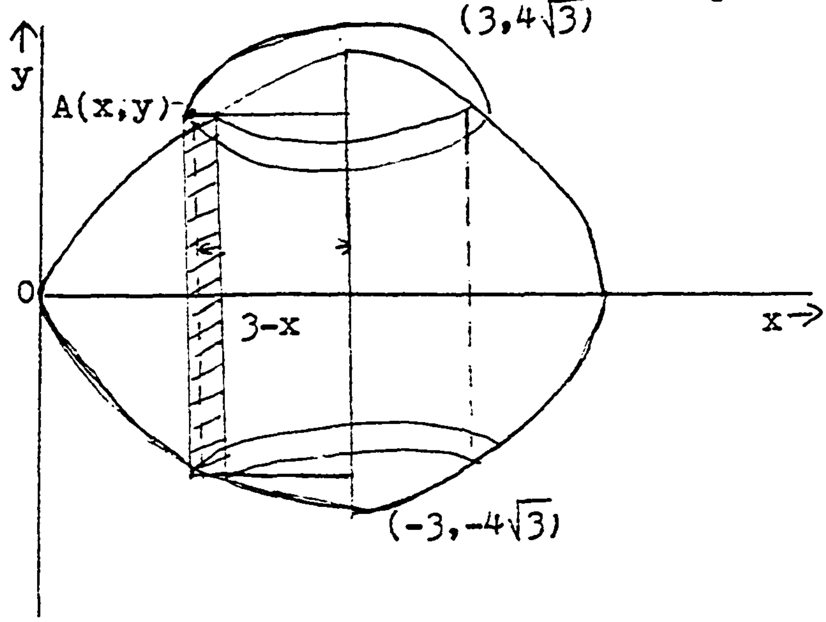
$$= 24\pi\ 3(1+(3/5)) = (192\sqrt{3}\pi)/5$$

The required volume is $192\sqrt{3}\pi/5$ cubic units.

3. Use the shell method in calculating the volume generated by revolving the area bounded by $16x = y^2$ and $x = 3$ about the line $x = 3$.

Solution:

Below is a sketch of the volume required



The height of the elementary strip is $2y =$ ⟶

The height of the elementary strip is $2y = 8\sqrt{x}$.

The average distance of the elementary strip from $x = 3$ is ⟶

The average distance of the elementary strip from $x = 3$ is $3-x$.

The volume of the cylindrical shell generated on rotating the representative rectangle about $x = 3$ is ⟶

The volume of the cylindrical shell generated on rotating the representative rectangle about $x = 3$ is $2\pi(3-x).\ 8\sqrt{x}\,\Delta x.$

The required volume is ⟶

The required volume is

$$16\pi \int_0^3 \sqrt{x}(3-x)\ dx$$

$$= 16\pi \int_0^3 (3x^{1/2}-x^{3/2})\ dx = 16\pi \left[2x^{3/2}-(2/5)x^{5/2}\right]_0^3$$

$$= 16\pi\ (2 3^{3/2}-(2/5)3^{5/2}) = 32\pi\ (3^{3/2}-3^{5/2}/5)$$

$$= 32\sqrt{3}\pi(3-(9/5)) = (192\sqrt{3}\pi)/5$$

The required volume is $192\sqrt{3}\pi/5$ cubic units.

T. Unit 20

Final Examination

The final examination consists of 25 questions that cover all the work done in mathematics 121 at Iowa State University.

Only one answer is correct in each question. Choose the correct answer from a, b, c, d or e.

1. Evaluate $\int_1^9 1/(x+3)\,dx$

   a. ln 9

   b. ln 4

   c. ln 12

   d. ln 3

   e. none of the above

2. Evaluate $\int_0^1 \sqrt{x}(1-\sqrt{x})\,dx$

   a. 4/9

   b. 4/11

   c. 4/13

   d. 4/15

   e. none of the above

3. If $y(x) \int_3^x g(t)\,dt$, find $y'(x)$.

   a. g(x)

   b. x g(x)

   c. 3x g(x)

d. $3\ g(x)$

e. none of the above

4. Find $\int(\cos^2 x - \sin^2 x)\ dx$

a. $-\sin 2x/2$

b. $\cos 2x + C$

c. $\cos 2x/2 + C$

d. $\sin 2x/2 + C$

e. none of the above

5. Find $\int (e^x + 7)^5\ e^x\ dx$

a. $e^x + 7 + C$

b. $(e^x + 7)^6/6 + C$

c. $(e^x + 7)^5 + C$

d. $e^{6x} + C$

e. none of the above

6. Find $\int \sec^2 x/(\tan x)\ dx$

a. $\ln|\tan x| + C$

b. $\ln|\tan x + \sec^2 x| + C$

c. $\ln|\sec^2 x| + C$

d. $\ln|\cos x| + C$

e. none of the above

7. Find $\int 1/(16x^2 + 25)\ dx$

a. $\tan 4x/5 + C$

b. arc tan $4x/5 + C$

c. ln $(16x^2 + 25) + C$

d. 1/4 arc tan $(4x/5) + C$

e. none of the above

8. Evaluate $\int \sqrt{x^2-25}\,dx$

a. $\ln\left|x + \sqrt{x^2+25}\right| + C$

b. $\sqrt{x^2-25} + C$

c. $\ln\left|x + \sqrt{x^2-25}\right| + C$

d. $\sin\sqrt{x^2-25} + C$

e. none of the above

9. Evaluate $\int 9x^2 \ln x\,dx$

a. $3x^3 \ln x - x^2 + C$

b. $9x^3 \ln x + x^2 + C$

c. $9x^3 \ln x + C$

d. $6x4 + C$

e. none of the above

10. Find $\int 2 \sin(\ln x)\,dx$

a. $\sin\ln x + 2\cos\ln x$

b. $x(\sin \ln x - \cos \ln x) + C$

c. $4x \cos\ln x + C$

d. $\sin \ln x + \cos\ln x$

e. none of the above

11. Complete the square of the function

$7x^2 + 3x + 4$

a. $7(x+3)^2+6$

b. $7(x+3)^2+1$

c. $7(x+2)^2+3/71$

d. $7\ (x+3/28)^2+103/28$

e. none of the above

12. Evaluate $\int 1/\ 4-(x-2)^2\ dx$

a. $2\arcsin\ (x-2)+C$

b. $\arcsin\ (x-2)+C$

c. $\arcsin\ (x-2)/2+C$

d. $\arcsin\ (x-2)/2+\ln 4x+C$

e. none of the above

13. Express $(5x+5)((x-1)(x^2+4))$ into partial fractions.

a. $2/(x-1)+(-2x+3)/(x^2\ \ 4)$

b. $2/(x-1)+6/(x^2+4)$

c. $1/x-1+(6x+3)/(x^2+4)$

d. $2/x-1+(7x+3)/(x^2+4)$

e. none of the above

14. Evaluate $\int (x+4)/(x+1)^2\ dx$

a. $1/(x+1)-7/(x+1)^2+C$

b. $\ln(x+1)-3/(x+1)+C$

c. $2\ln(x+1)+C$

d. $3/(x+1)-9(x+1)^2+C$

e. none of the above

15. Find $\int 70\cos^4 2x \sin^3 2x\, dx$

a. $35\cos^5 2x+7\sin^3 2x+C$

b. $-7\cos^5 2x+5\cos^7 2x+C$

c. $8\cos^4 2x+3\sin 2x+C$

d. $6\cos^5 2x+9\sin 2x+C$

e. none of the above

16. Find $\int 2/(5+3\sin x)\, dx$

a. arc tan $(5\tan x/2+3)/4+C$

b. arc tan $(5\tan x/2+2)/2+C$

c. arc cos $(3\tan x/2+1)+C$

d. arc sec$(5\tan x/2+3)+C$

e. none of the above

17. Evaluate $\int_0^4 \sqrt{t^2+6t+9}\, dt$

a. 18

b. 20

c. 22

d. 24

e. none of the above

18. Compute the area under the graph $(x-1)^2$ between

$x = 1$ and $x = 4$.

a. 7

b. 8

c. 9

d. 10

e. none of the above

19. Find the average value of $(\pi \cos x)/3\sqrt{\sin x}$ over the interval $[\pi/6, \pi/2]$.

a. $2-\sqrt{2}$

b. $2+\sqrt{2}$

c. $-2-\sqrt{2}$

d. $-2+\sqrt{2}$

e. none of the above

20. What is the area under the curve $y = \sin^2 x$ from $x = 0$ to $x = \infty$?

a. 70

b. 62

c. 34

d. infinite

e. none of the above

21. Express $(-2\sqrt{3}, 2)$ in polar coordinates.

a. $(4, \pi/6)$

b. $(3, \pi/3)$

c. $(4, 5\pi/6)$

d. $(3, \pi/6)$

e. none of the above

22. Does the sequence defined by $a_n = 5^n/n!$ converge or diverge?

a. converges to 0

b. diverges

c. converges to 25

d. converges to 225

e. none of the above

23. Find the area bounded by the curve $r^2 = 81 \cos 2\theta$.

a. 9

b. 81

c. 27

d. 243

e. none of the above

24. Find the area of the surface of revolution generated by revolving a loop of the curve $128y^2 = 16x^2 - x^4$ about the x axis.

a. 3π

b. 4π

c. 5π

d. 6π

e. none of the above.

25. Find the volume generated by revolving the ellipse $4x^2+9y^2=36$ about the y axis.

a. 12π

b. 18π

c. 24π

d. 30π

e. none of the above

XI. APPENDIX B: SIGN-ON PROCEDURE

PLATO is a general purpose computer which gives the user a lot of control. To operate the computer the student and computer will respond in the following way:

PLATO:

Press NEXT to begin

Student:

NEXT

PLATO:

Day, month, year

Welcome to PLATO

Type your name, then press NEXT

Student:

>agbor

PLATO:

Type the name of your PLATO group. Then, while holding the SHIFT key, press the STOP key. When you are ready to leave, you should press these same keys (SHIFT-STOP) to "sign off".

Student:

> ames

PLATO:

Type your password, then press NEXT

Student:

> mbi

PLATO:

AUTHOR MODE

Choose a lesson

HELP available

Student:

> agbor

PLATO:

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Choose a subject

BACK to exit, HELP is available

A. The PLATO Keyboard

Every PLATO terminal has a keyboard like a typewriter with special features. The following are special keys of the PLATO keyboard and their functions:

1. The HELP key allows students to make optional sections of a lesson.
2. The SHIFT key produces capital letters when a letter that is not a capital letter is pressed.
3. The ERASE key erases what has been typed.
4. The TAB key is equivalent to hitting the space bar as many times as is necessary to reach a preset column on the screen.
5. The NEXT key makes it possible to proceed to the next display.
6. The EDIT key is used for correcting typing.

7. The ANS key can be used by the student to get the correct answer to a question.
8. The HELP key also enables the user to enter the sequence.
9. The STOP key throws out output destined for the terminal.
10. The BACK key is used to review sequences.

B. Basics Aspects of PLATO

The PLATO interactive educational system consists of a repeating sequence which is a display on the student's screen followed by the student's response to the display. The display information consists of line drawings, graphs and animations. The student responds to this display by pressing a single key, by pointing at a particular area of the screen, by typing a word, sentence or mathematical expression or even by making a geometrical construction. Authors generally provide enough details about the possible student responses.